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COMPUTATION OF THE INCOMPLETE GAMMA FUNCTION RATIOS

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A method, made up of several algorithms, is given for computing the incomplete gamma function ratio, P(a, x) and its complement Q(a, x) for all real arguments $a > 0$, $x \geqslant 0$. The difficult case when a and x are large is treated by a modified version of Takenaga's method. The resulting computer program is efficient, yields both P and Q correctly to within 1 unit in the twelfth significant digit, or, at the user's option, to within one unit in the sixth or third significant digit. A FORTRAN listing of the program is included.

FOREWORD

The work covered in this report was done in the Science and Mathematics Research Group of the Warfare Analysis Department at the request of Dr. Marlin A. Thomas, Head of the Mathematical Statistics and Systems Simulation Branch.

The authors are indebted to Alfred H. Morris for carrying out a number of calculations for them using his algebraic routine, "Flap." The authors are also indebted to Michael P. Saizan for computing some of the coefficients given in Appendix C.

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ABSTRACT

A method, made up of several algorithms, is given for computing the incomplete gamma function ratio, $P(a, x)$ and its complement $Q(a, x)$ for all real arguments $a > 0$, $x \geqslant 0$. The difficult case when a and x are large is treated by a modified version of Takenaga's method. The resulting computer program is efficient, yields both P and Q correctly to within 1 unit in the twelfth significant digit, or, at the user's option, to within one unit in the sixth or third significant digit. A FORTRAN listing of the program is included.

TABLE OF CONTENTS

1. INTRODUCTION

The object of this report is to describe an efficient procedure for computing the incomplete gamma function ratio

$$P(a, x) = \gamma(a, x)/\Gamma(a), \quad a > 0, \quad x \geqslant 0, \tag{1}$$

and its complement

$$Q(a, x) = \Gamma(a, x)/\Gamma(a), \quad a > 0, \quad x \geqslant 0, \tag{2}$$

to within a unit in twelve, six, or three significant digits, where

$$\gamma(a, x) = \int_0^x e^{-t}\, t^{a-1}\, dt, \tag{3}$$

$$\Gamma(a, x) = \int_x^\infty e^{-t}\, t^{a-1}\, dt, \tag{4}$$

$$\gamma(a, \infty) = \Gamma(a, 0) = \Gamma(a) = \int_0^\infty e^{-t}\, t^{a-1}\, dt \tag{5}$$

with

$$0 \leqslant P \leqslant 1, 0 \leqslant Q = 1 - P \leqslant 1.$$

The computing program, in FORTRAN, designed for the CDC-6700*, uses 7 different mathematical formulations for P and Q, the asymptotic expansion ($a \to \infty$) for the complete gamma function, $\Gamma(a)$, and a series expansion for its reciprocal. They are all given in the next section with their domains of application. The algorithm used with each formulation is given in Section 5.

If the user desires 12 significant digits for P, Q, or both, a parameter ϵ is set, in the call to the program, to the value

$$\epsilon = \epsilon_1 = 5 \times 10^{-13} = 5(-13) \tag{6}$$

*The CDC-6700 is a large scale binary computer capable of one million arithmetic operations per second. It has a 60 bit binary word length of which 48 are used to express the mantissa of a number.

If, on the other hand, the user wants only 6 or 3 significant figures, then ϵ is set, respectively, to

$$\epsilon = \epsilon_2 = 5 \times 10^{-7} = 5(-7), \quad \epsilon = \epsilon_3 = 5 \times 10^{-4}.^{*} \tag{7}$$

Besides three available levels of accuracy for the output quantities P and Q, the program also has the capability to determine, in most cases, if the input quantities a and x are contained in "zones of non-computation." These are domains in the ax-plane for which $P \leq \epsilon$ or $Q \leq \epsilon$; for such (a, x), P is set to zero and Q is set to one or vice versa depending on the values of (a, x). Section 3 deals with the ways in which the program senses on the above inequalities.

The average computing time per case, when $\epsilon = \epsilon_2$ (6 significant digit accuracy in P and Q) is about 1 millisecond; for $\epsilon = \epsilon_1$ (12 significant digits in P and Q) the average computer time per case is about 15 to 20% longer and for $\epsilon = \epsilon_3$ about 10 to 15% shorter.

The validation of the program is described in Section 5. A complete FORTRAN IV listing is given in Appendix F.

There are many applications for the incomplete gamma function ratios such as computing the Poisson distribution, [1, p. 959]. Perhaps their most well-known use is for computing the Chi-square distributions $P(\chi^2|\nu)$ and $Q(\chi^2|\nu) = 1 - P(\chi^2|\nu)$, where

$$P(\chi^2|\nu) = \left[2^{\nu/2}\,\Gamma(\nu/2)\right]^{-1} \int_0^{\chi^2} e^{-u/2}\, u^{(\nu/2)-1}\, du. \tag{8}$$

By letting u = 2t in (8), the relationship between (χ^2, ν) and (x, a) becomes evident, i.e.,

$$P(\chi^2|\nu) = \int_0^{\chi^2/2} e^{-t}\, t^{(\nu/2)-1}\, dt/\Gamma(\nu/2) = P(\nu/2, \chi^2/2). \tag{9}$$

*More precisely, one should state that in the case of (6), P and Q are given correctly to within one unit in the twelfth significant digit, and to within one unit in the sixth or third significant digit in the case of (7).

so that

$$a = \nu/2, \quad x = \chi^2/2 .$$

The extensive use of the Chi-square distributions in statistics is sufficient to warrant a good machine program for their equivalent, the incomplete gamma function ratios. Surprisingly, to the best of our knowledge, there is only one computer program documented in the literature, [11]. In any case, the authors are not aware of a program for P and Q that has the overall speed, versatility, accuracy, and range of the present program.

2. MATHEMATICAL FORMULATIONS AND DOMAINS OF APPLICATION

In this section we summarize the mathematical formulations and state the domains on which they are used.

It is worth noting, since we require relative accuracy, that it is necessary to compute Q(P) if P(Q) is near one (and then find P(Q) from 1 − Q(P)). Consequently, the program is designed to compute P if $a \geq 1$ and $a \geq x$, because in this case P cannot be near one. Similarly, if $1 < a < x$, $x > \ln 10$, then Q is computed for it cannot be near one. If, however, $a < 1$ and $x \leq a$ then P may be near one and Q should be computed, although if x is sufficiently small P will, of course, be near zero and should be computed. This behavior of P near the origin of the ax-plane results from the nonexistence of $\lim\limits_{\substack{x \to 0 \\ a \to 0}} P(a, x)$, a fact easily concluded from (10) below.

Figure 1 shows the domains over which the various formulations are used. It and the flowchart on page 44 complement the remarks that follow. It is assumed $P > \epsilon$ and $Q > \epsilon$.

We proceed to specify the formulations and their domains of application.

$$P = \frac{R(a, x)}{a} \left[1 + \sum_{k=1}^{\infty} \frac{x^k}{(a+1) \cdots (a+k)} \right], \quad [1, \text{p. } 263] \tag{10}$$

$$R(a, x) = e^{-x} x^a / \Gamma(a) \tag{11}$$

<u>$a < 100$</u>

1) $1 \leq a$, $a \geq x$.

2) 2a is not an integer when $x > a$

 a) $1 \leq a < x \leq \ln 10$

 b) $a < 1$, $x < 1.5$, $P \leq 0.90$

 c) $a < 1$, $x \geq 1.5$, $R > 0.101\, x[(x + 2 - a)/(x + 1)]$

<u>$a \geq 100$</u> $3a > 4x$

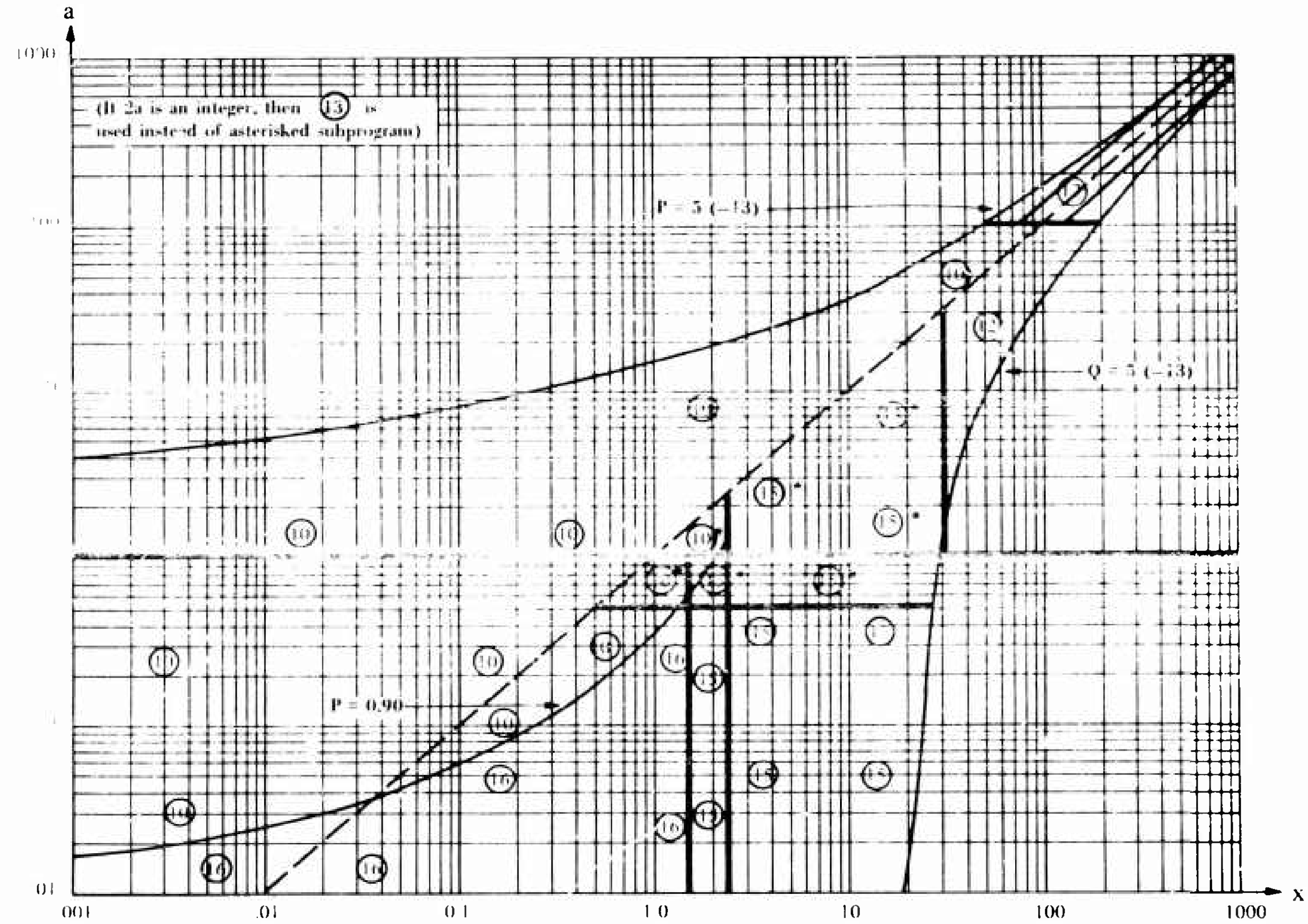

Figure 1. Domains of Calculation Showing Subprogram Used

$$Q(a, x) = \frac{R(a, x)}{x}\left[1 + \sum_{k=1}^{N} \frac{(a-1)\cdots(a-k)}{x^k}\right], \qquad [1, p. 263] \qquad (12)$$

<u>$a < 100$</u>

$1 < a < x, \quad x \geq 31(\epsilon = \epsilon_1), \quad x \geq 17(\epsilon = \epsilon_2), \quad x \geq 9.7(\epsilon = \epsilon_3)$*

<u>$a \geq 100$</u> $\quad 4x > 5a$

$$Q(k + g, x) = Q(k + g - 1, x) + R(k + g - 1, x)/(k + g - 1), \qquad [1, p. 262] \qquad (13)$$

$$g = \begin{cases} 1 & a \text{ is an integer} \\ 1/2 & a \text{ is not an integer; } k = 1, 2, \cdots, a - g. \end{cases}$$

$$Q(g, x) = \begin{cases} (2/\sqrt{\pi})\displaystyle\int_{\sqrt{x}}^{\infty} e^{-u^2}\,du = \operatorname{erfc}(\sqrt{x}), & g = 1/2 \\ e^{-x}, & g = 1. \end{cases} \qquad (14)$$

<u>$a < 100$</u>

$2a$ is an integer, $a \leq x < 31, 17, 9.7$ for $\epsilon = \epsilon_1, \epsilon_2, \epsilon_3$, respectively.

$$Q(a, x) = R(a, x)\left[\cfrac{1}{x + \cfrac{1 - a}{1 + \cfrac{1}{x + \cfrac{2 - a}{1 + \cfrac{2}{x + \cfrac{3 - a}{1 + \cfrac{3}{x + \ddots}}}}}}}\right]. \qquad [10, p. 356] \qquad (15)$$

<u>$a < 100$</u>

1. $2a$ is not an integer
   a) $1 < a < x$, in $10 < x < 31, 17, 9.7$ for $\epsilon = \epsilon_1, \epsilon_2, \epsilon_3$, respectively.
   b) $a < 1, x \geq 1.5, R \leq 0.101\, x(x + 2 - a)/(x + 1)$

*See (6) and (7) for values of $\epsilon_1, \epsilon_2, \epsilon_3$.

$$Q(a, x) = J - (H + L) + J(H + L) - HL + JHL \tag{16}$$

$$J = -a \sum_{k=1}^{\infty} \frac{(-x)^k}{(a+k)\,k!}$$

$$L = \sum_{k=1}^{\infty} \frac{(a \ln x)^k}{k!}$$

$$H = \begin{cases} \displaystyle\sum_{k=2}^{\infty} C_k\, a^{k-1} & 0 < a < 1/2 \\[2ex] (1/a)\left[(1-a) + \displaystyle\sum_{k=2}^{\infty} C_k (a-1)^{k-1}\right] & 1/2 \leqslant a < 1. \end{cases}$$

The coefficients C_k are given in Appendix A, also see (25). Discussion of (16) given below and on pages 10 and 11.

<u>$a < 100$</u>

$a < 1, \; x < 1.5, \; P > 0.90.$

The need for (16) arises when $a < 1$, since then it is not true, even though $x < a$, that P cannot be near one. In fact, for $a = x \cong 0.038$, $P \cong 0.90$, for $a = 10^{-3}$, $x = 10^{-4}$, $P = 0.9913 \cdots$, and for $a = 10^{-5}$, $x = 10^{-10}$, $P = 0.9997 \cdots$. Thus, in such cases it is Q, rather than P, which must be computed. This behavior of P near the origin of the ax-plane, as mentioned previously, can be accounted for by noting from (10) that the double limit for P as $x \to 0$, $a \to 0$ does not exist.

If $a \geqslant 100$ and $3a \leqslant 4x \leqslant 5a$, none of the above formulations suffice for computing purposes. (If $4x < 3a$, (10) is used and if $4x > 5a$ then (12) is used.) In this main area of difficulty, we have found that Takenaga's method [9], with appropriate changes, works quite well. The final formulas are given by (17) – (24). The mathematical formalism is easily programmed into an accurate and relatively efficient procedure for computing $P(a, x)$ when $x \leqslant a - 1/3$, or $Q(a, x)$ when $x > a - 1/3$. The details for deriving (17) – (24) are given in Section 4.

Let

$$
T(a, x) = \begin{cases} P(a, x) & \text{if } x \leqslant a - 1/3 \\ Q(a, x) & \text{if } x > a - 1/3 . \end{cases}
$$

Then Takenaga's analysis, after some changes, reduces to

$$
\begin{aligned}
T(a, x) \cong C \Bigg[B_0 & - \frac{1}{12a^2} B_2 - \frac{1}{18a^4} \left(B_3 - \frac{1}{16} B_4 \right) \\
& - \frac{1}{24a^6} \left(B_4 - \frac{37}{225} B_5 + \frac{1}{432} B_6 \right) \\
& - \frac{1}{30a^8} \left(B_5 - \frac{743}{3024} B_6 + \frac{49}{4320} B_7 - \frac{5}{82944} B_8 \right) \\
& \vdots \\
& \frac{1}{a^{24}} \left(\frac{1}{78} B_{13} - \cdots - 2.3414\,67226\,51062 \times 10^{-22} B_{24} \right) \\
& + \frac{A_2}{15a^3} + \frac{1}{3a^5} \left(\frac{A_3}{7} - \frac{A_4}{60} \right) + \frac{1}{27a^7} \left(A_4 - \frac{29}{140} A_5 + \frac{1}{160} A_6 \right) \\
& + \frac{1}{a^9} \left(\frac{1}{33} A_5 - \frac{193}{22680} A_6 + \frac{1187}{2268000} A_7 - \frac{1}{155520} A_8 \right) \\
& \vdots \\
& + \frac{1}{a^{25}} \left(\frac{1}{81} A_{13} - \cdots - 2.2478\,08537\,45020 \times 10^{-21} A_{24} \right) \Bigg]
\end{aligned} \tag{17}^*
$$

*The numerical coefficients in (17) and (18) were computed by Alfred H. Morris using his "Flap" algebraic routine, [5]. The complete computed sets of coefficients are given in Appendix B.

where, with $b = a - 1$

$$C = 1 + \frac{1}{36} b^{-1} - \frac{31}{2592} b^{-2} + \frac{3413}{1399680} b^{-3} + \frac{361733}{201553920} b^{-4} \tag{18}$$

$$- \frac{113888281}{50791587840} b^{-5} + \frac{7565202533}{7836416409600} b^{-6} - \cdots,$$

$$\alpha = 3(a - 1/3)^{1/2} \tag{19}$$

The A_k and B_k, for a given integer k, have different meanings depending on whether $x < a - 1/3$ or $x > a - 1/3$. If $x \leq a - 1/3$, then

$$A_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} z^{2k+1} e^{-z^2/2} \, dz \tag{20}$$

$$B_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} z^{2k} e^{-z^2/2} \, dz, \qquad k = 0, 1, \cdots, \tag{21}$$

where

$$s = \alpha \left[\left(\frac{x}{a - 1/3} \right)^{1/3} - 1 \right] \leq 0. \tag{22}$$

If $x > a - 1/3$, then (20) and (21) are replaced by

$$A_k = \frac{1}{\sqrt{2\pi}} \int_{s}^{\infty} z^{2k+1} e^{-z^2/2} \, dz \tag{23}$$

$$B_k = \frac{1}{\sqrt{2\pi}} \int_{s}^{\infty} z^{2k} e^{-z^2/2} \, dz, \qquad k = 0, 1, 2, \cdots. \tag{24}$$

In this case also, s is given by (22), but now it is positive. We remark that the program uses a slightly different form for (17). See (115).

The evaluation of $R(a, x)$ (see (11)), which contains $1/\Gamma(a)$, is carried out in two different ways. If $a < 30$, then the factor $x^a e^{-x}$ is computed in the form $e^{-x + a \ln x}$, and subsequently $\Gamma(a)$ is evaluated as a separate factor by the recurrence relation

$$\Gamma(j + \lambda) = (j + \lambda - 1)\, \Gamma(j + \lambda - 1), \quad j = 1, 2, \cdots, a - \lambda, \quad \lambda \neq 0,$$

where λ denotes the fractional part of a and $a > 1$. The procedure is initiated by computing $\Gamma(\lambda)$ from a series expansion for its reciprocal. The quantity $1/\Gamma(\lambda)$ is computed to 14 significant digits by the polynomial approximations, [13],

$$1/\Gamma(\lambda) = \begin{cases} \lambda \left[1 + \sum_{k=2}^{17} C_k \lambda^{k-1} \right], & 0 < \lambda < 1/2 \\ \left[1 + \sum_{k=2}^{17} C_k (\lambda - 1)^{k-1} \right], & 1/2 \leqslant \lambda \leqslant 1; \end{cases} \tag{25}$$

the C_k's are given in Appendix A. By using the first equation of (25) and $\Gamma(\lambda) = \Gamma(1 + y) = y\,\Gamma(y)$ where $\lambda = 1 + y$, so that $1/2 < \lambda \leqslant 1$, $-1/2 < y \leqslant 0$, the second equation of (25) is obtained. If $\lambda = 0$, then the above recurrence relation is started with $j = 2$ (instead of one) and $\Gamma(1) = 1$.

If $30 < a$, then the asymptotic expansion for $\ln \Gamma(a)$, $(a \to \infty)$, [1, p. 257],

$$\ln \Gamma(a) \simeq (a - 1/2) \ln a - a + \frac{1}{2} \ln 2\pi + L(a), \tag{26}$$

where

$$L(a) = \frac{1}{12a} - \frac{1}{360a^3} + \frac{1}{1260a^5} - \frac{1}{1680a^7}, \quad \left(\sim \frac{\theta}{12}, 0 < \theta < 1\right) \tag{27}$$

is combined with the logarithm of the numerator of (11) to give, after exponentiation,

$$R(a, x) \cong \frac{1}{\sqrt{2\pi}} \sqrt{a} \exp \left[(a - x) + a \ln \frac{x}{a} - L(a) \right] \tag{28}$$

It is easy to show that the argument of the exponential in (28) is always negative for admissible values of a and x. Hence no scaling problems can occur due to the large magnitude of a and x.

We do not give detailed derivations of the above equations except for (17) – (24) which are developed in Section 4. Equations (10), (12), and (13) are easily obtained using integration by parts on (1) or (2); (14) follows directly from (2) when $a = 1/2$ and $t = u^2$; (15) is derived in [10]. Equation (16) follows by substituting

$$P = \left[x^a/\Gamma(a+1)\right]\left[1 + a\sum_{k=1}^{\infty}\frac{(-x)^k}{(a+k)\,k!}\right] \quad \text{(See [1, p. 262])} \tag{29}$$

$$x^a = 1 + \sum_{k=1}^{\infty}\frac{(a\,\ln x)^k}{k!} \tag{30}$$

$$1/\Gamma(a+1) = \begin{cases} 1 + \sum_{k=2}^{\infty} C_k\, a^{k-1}, & 0 < a < 1/2, \text{ (See (25))} \\ \dfrac{1}{a}\left[1 + \sum_{k=2}^{\infty} C_k (a-1)^{k-1}\right], & 1/2 \leq a < 1. \end{cases} \tag{31}$$

in the expression $1 - P\ (= Q)$. The derivation of the series expansion for $1/\Gamma(a)$ is sketched in [13]. The asymptotic expansion for $\ln \Gamma(a)$ is discussed in [6, p. 293].

Table 1 has been compiled to give some idea of the number of terms of (10), (12), or (15) that are needed for various arguments (a, x) with $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$. The third column of the table identifies the equation that was used by its number in the text. The fourth and fifth columns list the number of terms of that equation needed for $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, respectively. For example a = 7.1, x = 28 uses 12 terms of (15) for ϵ_1 and 7 terms of (12) for ϵ_2. No examples are given for (13) since the number of iterations is always the integer part of a. Also no data is given for (16). Since (16) is used when a < 1 and x < 1.5, it yields an efficient algorithm. In the worse case, when $x \geq 1.5$ and $\epsilon = \epsilon_1$, no more than 18 terms of J are needed; the evaluation of L requires less than 12 terms and in the worse case of $a \geq 1/2$, no more than 17 terms of H are used.

In the next section we show how, for small $\epsilon > 0$, advantage is taken of a property of P and Q. Namely, if a is large, then x cannot be too far from a, otherwise $P < \epsilon$ or $1 - P = Q < \epsilon$ depending on whether a > x, or x > a, respectively. In other words, there is a very limited region of the ax-plane for which $\epsilon < P < 1 - \epsilon$ and for which extensive calculations, using one of the above formulations, is necessary. Inequalities will be derived which, in the case of equality, give very close approximations to the boundaries of this region. Earlier, we referred to the exterior and boundary of this region as the "zone of non-computation." The computing time for any one of the inequalities is relatively short. As a result, the average computing time per case is markedly reduced, since the average computation time for computing P and Q, when (a, x) is in the "zone of computation" ($\epsilon < P < 1 - \epsilon$) is generally much longer than when it is not.

Table 1. Number of Terms used by Subprograms for Some (a, x)

a	x	Eq.	No. of Terms ϵ_1	No. of Terms ϵ_2	a	x	Eq.	No. of Terms ϵ_1	No. of Terms ϵ_2
.4	.1	10	8	5	15.1	2	10 / *	13	*
	.4	10	12	7		10	10	29	18
	1	10	16	10		16	15	28	20
	5	15	32	6		23	15 / 12	24	13
	10	15	20	4		30	15 / 12	20	12
	15	15 / *	16	*		35	12	15	11
	21	15 / *	12	*		40	12	15	10
2.1	.1	10	8	5	30.1	6	10 / *	16	*
	2.1	10	19	12		15	10	28	17
	7	15	20	3		20	10	35	22
	13	15	14	2		30	10	51	35
	18	15 / *	12	*		32	12	29	23
	25	15 / *	10	*		50	12	25	17
	32	12 / *	12	*		80	12 / *	20	*
7.1	.5	10	10	6	99.99	50	10 / *	35	*
	2	10	15	9		75	10	56	35
	7.1	10	29	19		99.9	10	86	61
	13	15	16	10		100	12	65	49
	20	15 / 12	14	7		120	12	54	37
	28	15 / 12	12	7		150	12	43	27
	32	12 / *	10	*		185	12 / *	35	*

*Indicates (a, x) not in "zone of computation" (see Section 3).

3. ZONE OF COMPUTATION

The function $P(a, x)$ ($Q(a, x)$) has the property that for a given $\epsilon > 0$, there exists a function $\underline{x}(a)$ ($\bar{x}(a)$) such that $P \leq \epsilon$ ($Q \leq \epsilon$) for all $x \leq \underline{x}(a)$ ($x \geq \bar{x}(a)$). Of course $\underline{x}$ and $\bar{x}$ are also functions of ϵ. Figure 2 shows the curves $\underline{x}(a)$ and $\bar{x}(a)$ on two scales for $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$. Referring to the figure, P or Q is computed using (10) – (28) if (a, x) is contained between $\underline{x}$ and $\bar{x}$. We call this region the "zone of computation." If (a, x) is asserted to be outside this region by the estimates given below for $\underline{x}$ and $\bar{x}$, then P is set to zero or one and Q to one or zero. As can be seen from Figure 2, the "zone of computation" makes up only a small part of the plane region $x \geq 0, a > 0$. Therefore, it is advantageous to obtain close estimates for $\underline{x}$ and $\bar{x}$ which can be used efficiently by the program.

Two different approaches are used to find two different sets of estimates for $\underline{x}$ and $\bar{x}$. Associated inequalities are given which can be easily incorporated into the program to effectively decide if (a, x) is not between $\underline{x}$ and $\bar{x}$.

Let the first estimates for $\underline{x}(a)$ and $\bar{x}(a)$ be denoted by $\underline{x}_1$ and $\bar{x}_1$, respectively. The expression for $\bar{x}_1$ is derived in [12]. It is given by

$$\bar{x}_1 = a\left(1 - \frac{1}{9a} + \bar{y}/\sqrt{9a}\right)^3, \quad a \geq 15, \tag{32}$$

where $\bar{y}$ satisfies the equation

$$\mathrm{erfc}(\bar{y}/\sqrt{2}) = 2\epsilon .$$

The constant $\bar{y}$ is determined once ϵ is specified. For example, when $\epsilon = \epsilon_1$, $\bar{y} = 7.1306 \cdots$; $\epsilon = \epsilon_2$, $\bar{y} = 4.892$; $\epsilon = \epsilon_3$, $\bar{y} = 3.29053$. A set of correction factors to improve (32) has been given in [8], and they show that $\bar{x}_1 > \bar{x}$. Therefore,

$$Q(a, x) < \epsilon \quad \text{for all } x \geq \bar{x}_1 > \bar{x} .$$

Similarly

$$\underline{x}_1 = a\left(1 - \frac{1}{9a} - \bar{y}/\sqrt{9a}\right)^3, \quad a \geq 15, \tag{33}$$

where again it can be established from [8] that $\underline{x}_1 < \underline{x}$. Therefore,

$$P(a, x) < \epsilon \quad \text{for all } x \leq \underline{x}_1 < \underline{x} .$$

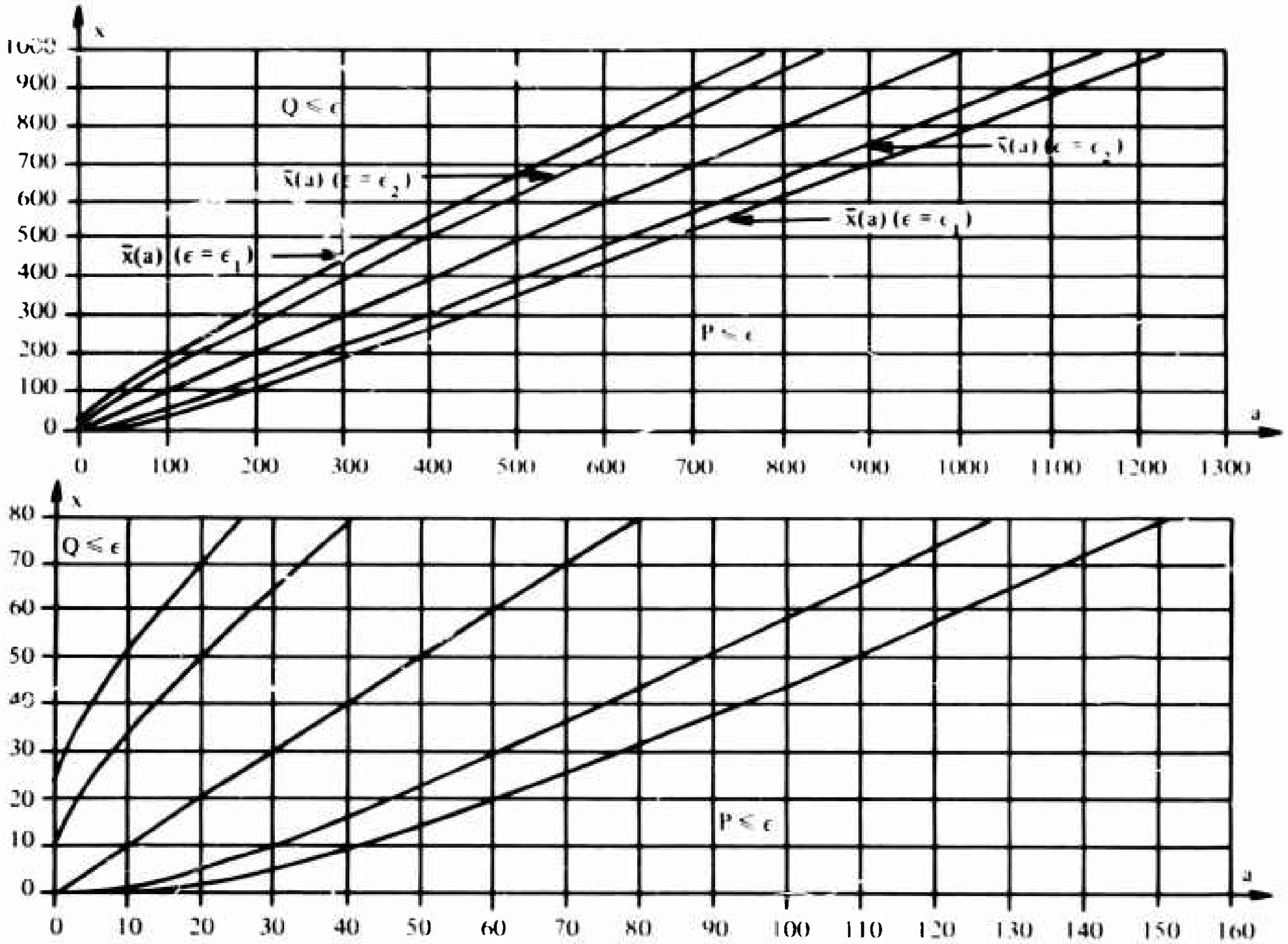

Figure 2. Zone of Computation, $\epsilon = \epsilon_1$, $\epsilon = \epsilon_2$

In order to use these results in the program, new functions $\overline{F}$ and $\underline{F}$ are defined. Using the fact that Q is a decreasing function of x and an increasing function of a, we conclude from (32), for given (a, x), that if

$$\overline{F} \equiv x - a\left(1 - \frac{1}{9a} + \overline{y}/\sqrt{9a}\right)^3 \geqslant 0\,, \quad a \geqslant 15 \tag{34}$$

then $Q < \epsilon$. In the same way, we conclude from (33), that if

$$\underline{F} \equiv x - a\left(1 - \frac{1}{9a} - \overline{y}/\sqrt{9a}\right)^3 \leqslant 0, \quad a \geqslant 15 \tag{35}$$

then $P < \epsilon$. Clearly the quantities $\overline{F}$ and $\underline{F}$ are cheap to compute, requiring only a square root and a few arithmetic operations. However, they are only useful if a is not small $(a \geqslant 15)$.

The other estimates for $\overline{x}$ and $\underline{x}$ are denoted by $\overline{x}_2$ and $\underline{x}_2$, respectively. We first show for fixed a and ϵ, that there exists a unique positive value of x, say $\underline{x}_2$, such that

$$R(a, \underline{x}_2) = a\,[1 - \underline{x}_2/(a+1)]\epsilon, \quad \underline{x}_2 < a, \quad \text{(See (11))}, \tag{36}$$

and

$$P(a, x) < \epsilon \quad \text{for all } x \leqslant \underline{x}_2 < a\,. \tag{37}$$

Then we show that if $\overline{x}_2$ is determined (uniquely) from

$$R(a, \overline{x}_2) = \overline{x}_2\,[(\overline{x}_2 + 2 - a)/(\overline{x}_2 + 1)]\epsilon, \quad 0 < a \leqslant 1, \quad \text{or } a \geqslant 2\,, \tag{38}$$

then

$$Q(a, x) \leqslant \epsilon \quad \text{for all } x \geqslant \overline{x}_2\,. \tag{39}$$

For $1 < a < 2$, it can be shown $Q(a, x) > \epsilon$ for all $x \leqslant \overline{x}_2$, therefore, we replace (38) by

$$R(a, \overline{x}_2) = (\overline{x}_2 + 1 - a)\epsilon \quad 1 \leqslant a < 2, \quad \overline{x}_2 > a \tag{40}$$

and show that (39) holds.

Let, for a given value of a and ϵ,

$$f(x) \equiv R(a, x) - a[1 - x/(a+1)]\,\epsilon, \qquad x < a.$$

Then for small $\epsilon > 0$, there exists a unique positive value of x which we identify as $\underline{x}_2$ such that $f(\underline{x}_2) = 0$, i.e., such that (36) is satisfied. Indeed, $f(0) < 0$, $f'(x) > 0$ for $x \leqslant a$, and $f(a) > 0$ for small $\epsilon > 0$. The last inequality on f follows from

$$f(a) = \frac{e^{-a}a^a}{\Gamma(a)} - \frac{a}{a+1}\,\epsilon = a\left[\frac{e^{-a}a^a}{\Gamma(a+1)} - \frac{\epsilon}{a+1}\right].$$

If $a \leqslant 1$, then $\Gamma(a+1) \leqslant 1$, and

$$f(a) > a\,[e^{-a}a^a - \epsilon] > a\,[e^{-1} - \epsilon] > 0, \qquad (\epsilon < 1/e \sim 0.368).$$

If $a \geqslant 1$, then using (26)

$$f(a) > \frac{e^{-a}a^a}{e^{-a}a^a\sqrt{2\pi/a}\,e^{1/12a}} - \epsilon = \sqrt{\frac{a}{2\pi}}\,e^{-1/12a} - \epsilon \tag{41}$$

where the inequality follows by noting that an asymptotic series with decreasing successive terms of alternating sign yields an approximation with an error less than the first neglected term and of the same sign. The right hand side of (41) for $a \geqslant 1$ takes its minimum value at $a = 1$, hence,

$$f(a) > \frac{1}{\sqrt{2\pi}}\,e^{-1/12} - \epsilon > 0, \qquad (\epsilon < 0.367).$$

Now using (10)

$$P(a,x) = \frac{R(a,x)}{a}\left[1 + \sum_{k=1}^{\infty}\frac{x^k}{(a+1)\cdots(a+k)}\right] \tag{42}$$

$$< \frac{R(a,x)}{a}\left[1 + \sum_{k=1}^{\infty}\left(\frac{x}{a+1}\right)^k\right]$$

$$= R(a,x)\Bigg/\left[a\left(1 - \frac{x}{a+1}\right)\right] = \xi(a,x)$$

Consequently, since ξ is an increasing function of x when $x \leqslant a$, (37) must hold, i.e.,

$$P(a, x) < \xi(x, a) \leqslant \xi(\underline{x}_2, a) = \epsilon, \quad x \leqslant \underline{x}_2 \leqslant a. \tag{43}$$

The following argument for the existence and uniqueness of $\bar{x}_2$ in (40) depends on the function

$$g(x) \equiv R(a, x) - (x + 1 - a)\epsilon, \quad x > a, \quad a \geqslant 1$$

where a and ϵ are fixed. The proof is similar to the one for (36). Clearly $g(\infty) = -\infty$, and it is easy to show $g'(x) < 0$ for $x \geqslant a$. The proof will be complete if it can be proved that $g(a) > 0$. We have, using (26) again with $a \geqslant 1$, that

$$g(a) = R(a, a) - \epsilon > \sqrt{\frac{a}{2\pi}}\, e^{-1/12a} - \epsilon.$$

The expression on the right also appeared in (41). Therefore, $g(a) > 0$ provided $(1/\sqrt{2\pi})\, e^{-1/12} > \epsilon$, i.e., for $\epsilon < 0.367$.

We proceed to show that (39) holds for (40). From (2), with $a - 1 < x$,

$$Q(a, x) = \frac{1}{\Gamma(a)} \int_x^\infty \exp\left[-t + (a - 1)\ln t\right] dt, \quad a \geqslant 1 \tag{44}$$

$$\leqslant \frac{1}{\Gamma(a)} \int_x^\infty \left| \frac{1 + (a - 1)/t}{-1 + (a - 1)/x} \right| \exp\left[-t + (a - 1)\ln t\right] dt$$

$$= \frac{e^{-x}\, x^a}{\Gamma(a)(x + 1 - a)} = \frac{R(a, x)}{(x + 1 - a)}, \quad x > a - 1 \geqslant 0.$$

It is easy to show, by differentiation, that $R(a, x)/(x + 1 - a)$ is a decreasing function of x for $x > a - 1 \geqslant 0$. Hence,

$$Q(a, x) \leqslant R(a, x)/(x + 1 - a) \leqslant R(a, \bar{x}_2)/(\bar{x}_2 + 1 - a) = \epsilon, \quad 0 \leqslant a \leqslant \bar{x}_2 \leqslant x. \tag{45}$$

Next, we establish the existence and uniqueness of $\bar{x}_2$ in (38). Consider the function

$$h(x) = \left[(x + 1)R/x(x + 2 - a)\right] - \epsilon, \quad 0 < a < 1, \quad \text{or } a \geqslant 2 \text{ and } x > (a - 2),$$

so that

$$h'(x) = [R/x(x+2-a)]\left[\frac{a-1}{x} + a - (x+1)(x+3-a)/(x+2-a)\right]$$

$$= -(R/x)\left[1 + \frac{(1-a)(2-a)}{x(x+2-a)^2}\right]$$

Now if $0 < a < 1$, then $h(0+) = \infty$, $h(\infty) = -\epsilon$ and $h'(x) < 0$. Hence, there exists a unique positive value of x, say $\bar{x}_2$ such that $h(\bar{x}_2) = 0$. Clearly this implies (38) is satisfied by $\bar{x}_2$ since $x + 2 - a > 0$.

If $a > 2$, $x > a - 2$, then $h(a-2+) = \infty$, $h(\infty) = -\epsilon$ and $h'(x) < 0$. Hence, by the same arguments as above, there exists a unique positive value of x, say $\bar{x}_2$ that satisfies (38).

For $0 < a < 1$, it is shown in [4] that $Q \leqslant h(x) + \epsilon$.

This is easily established here by considering

$$M(x) \equiv Q(a, x) - (h(x) + \epsilon)$$

and

$$M'(x) = (1-a)(2-a)\, R(a, x)/[x(x+2-a)]^2$$

Then if $0 < a \leqslant 1$, $M(0+) = -\infty$, $M(\infty) = 0$, $M'(x) > 0$. Consequently $M(x) < 0$ for $0 < a < 1$. Similarly for $a > 2$, $x > a - 2$, $M(a-2+) = -\infty$, $M(\infty) = 0$, $M'(x) > 0$ so that $M(x) < 0$ for $a > 2$, $x > a - 2$ (note also that $M(x) = 0$ for $a = 1$ and $a = 2$). Finally, since $h'(x) < 0$, $h(x) + \epsilon$ is a decreasing function of x, so that for all $x \geqslant \bar{x}_2$,

$$Q(a, x) \leqslant (x+1)\, R/[x(x+2-a)] \leqslant (\bar{x}_2 + 1)\, R(\bar{x}_2, a)/[\bar{x}_2(\bar{x}_2 + 2 - a)] = \epsilon \quad (46)$$

This establishes (39) for $\bar{x}_2$ satisfying (38) and includes the values $a = 1$, $a = 2$. Incidently, $h(x) + \epsilon$ can be obtained as an approximation for Q from (98) with $n = 1$.

In [6, p. 70], the first inequality in (45) is derived in a different way. The inequalities (34) and (35) can be sharpened [8]. The same can be said for (43), (45), [3], and (46), [4]. We did not feel that sharper estimates at the expense of more computing time were justified since the above estimates were already very good as can be seen by looking at Table 2. The table also shows that (34) and (35) are better than (43) and (46) for extremely large a.

Table 2. Comparison of $\underline{x}_i(a)$, $\bar{x}_i(a)$ with $\underline{x}(a)$, $\bar{x}(a)$, i = 1,2

$\underline{x}_2$	$\underline{x}_1$	$\underline{x}$	a	$P(a, \underline{x}_2)$	$P(a, \underline{x}_1)$	$P(a, \underline{x})$
15.010	14.91	15.019	37.95	4.93 (−7)	4.21 (−7)	5 (−7) = ϵ_2
25.012	24.92	25.023	53.53	4.94 (−7)	4.43 (−7)	5 (−7) = ϵ_2
50.018	49.95	50.038	88.75	4.92 (−7)	4.66 (−7)	5 (−7) = ϵ_2
75.021	74.97	75.049	121.57	4.91 (−7)	4.75 (−7)	5 (−7) = ϵ_2
100.018	99.99	100.055	153.15	4.89 (−7)	4.82 (−7)	5 (−7) = ϵ_2
250.03	250.05	250.102	331.68	4.88 (−7)	4.91 (−7)	5 (−7) = ϵ_2
500.02	500.10	500.147	613.79	4.85 (−7)	4.95 (−7)	5 (−7) = ϵ_2
1000.03	1000.18	1000.218	1159.19	4.85 (−7)	4.97 (−7)	5 (−7) = ϵ_2
5000.04	5000.48	5000.514	5350.72	4.85 (−7)	4.99 (−7)	5 (−7) = ϵ_2
10000.02	10000.67	10000.715	10494.2	4.83 (−7)	4.99 (−7)	5 (−7) = ϵ_2
15.001	14.73	15.003	50.49	4.98 (−13)	2.58 (−13)	5 (−13) = ϵ_1
25.001	24.75	25.004	68.69	4.97 (−13)	3.18 (−13)	5 (−13) = ϵ_1
50.004	49.78	50.009	108.65	4.97 (−13)	3.80 (−13)	5 (−13) = ϵ_1
75.006	74.80	75.015	145.08	4.96 (−13)	4.08 (−13)	5 (−13) = ϵ_1
100.005	99.82	100.015	179.69	4.96 (−13)	4.27 (−13)	5 (−13) = ϵ_1
250.007	249.88	250.029	371.29	4.95 (−13)	4.64 (−13)	5 (−13) = ϵ_1
500.01	499.95	500.048	668.09	4.94 (−13)	4.80 (−13)	5 (−13) = ϵ_1
1000.01	999.98	1000.074	1234.22	4.92 (−13)	4.89 (−13)	5 (−13) = ϵ_1
5000.01	5000.10	5000.173	5513.12	4.92 (−13)	4.96 (−13)	5 (−13) = ϵ_1
10000.04	10000.30	10000.380	10722.2	4.92 (−13)	4.97 (−13)	5 (−13) = ϵ_1

Table 2. Comparison of $\underline{x}_i(a)$, $\bar{x}_i(a)$ with $\underline{x}(a)$, $\bar{x}(a)$, i = 1, 2. (Continued)

$\bar{x}_2$	$\bar{x}_1$	$\bar{x}$	a	$Q(a, \bar{x}_2)$	$Q(a, \bar{x}_1)$	$Q(a, \bar{x})$
** 14.995	16.38	14.994	1.15	4.99 (−7)	1.27 (−7)	5 (−7) = ϵ_2
24.992	25.58	24.989	5.36	4.99 (−7)	3.05 (−7)	5 (−7) = ϵ_2
49.99	50.25	49.979	20.03	4.96 (−7)	4.21 (−7)	5 (−7) = ϵ_2
74.975	75.15	74.957	37.17	4.95 (−7)	4.51 (−7)	5 (−7) = ϵ_2
99.975	100.10	99.948	55.57	4.94 (−7)	4.66 (−7)	5 (−7) = ϵ_2
249.975	249.99	249.906	177.02	4.90 (−7)	4.87 (−7)	5 (−7) = ϵ_2
499.975	499.92	499.860	394.90	4.88 (−7)	4.93 (−7)	5 (−7) = ϵ_2
999.97	999.83	999.781	849.49	4.85 (−7)	4.96 (−7)	5 (−7) = ϵ_2
4999.99	4999.54	4999.497	4657.97	4.83 (−7)	4.98 (−7)	5 (−7) = ϵ_2
9999.97	9999.32	9999.277	9514.47	4.83 (−7)	4.99 (−7)	5 (−7) = ϵ_2
15	*	14.998	2.60 (−5)	4.99 (−13)	NA	5 (−13) = ϵ_1
25	36.73	24.999	0.3075	4.99 (−13)	3.10 (−18)	5 (−13) = ϵ_1
50	51.67	49.999	9.46	4.99 (−13)	1.23 (−13)	5 (−13) = ϵ_1
74.994	75.95	74.991	22.85	4.99 (−13)	1.39 (−13)	5 (−13) = ϵ_1
99.985	100.68	99.980	38.15	4.98 (−13)	3.20 (−13)	5 (−13) = ϵ_1
249.99	250.29	249.970	146.42	4.96 (−13)	4.36 (−13)	5 (−13) = ϵ_1
499.985	500.15	499.951	349.58	4.95 (−13)	4.70 (−13)	5 (−13) = ϵ_1
999.98	1000.06	999.924	783.43	4.94 (−13)	4.85 (−13)	5 (−13) = ϵ_1
4999.99	4999.92	4999.831	4504.52	4.92 (−13)	4.96 (−13)	5 (−13) = ϵ_1
9999.99	9999.83	9999.751	9295.57	4.91 (−13)	4.97 (−13)	5 (−13) = ϵ_1

*(34) not applicable for a = 2.60 (−5) = 2.60×10^{-5}.

**$\bar{x}_2$ computed from (40) instead of (38).

As stated before, the use of (34), (35), (43), (45) and (46) contribute significantly to the overall efficiency of the program. Moreover, the computation of $R(a, x)$ in (43), (45) or (46) is not always wasted if these inequalities are not satisfied, because this function is also needed for computing P from (10) or Q from (12) or (15).

In the next section we give a somewhat detailed discussion of Takenaga's method, for use when $a \geqslant 100$, and the changes we made to it.

4. TAKENAGA'S METHOD AND MODIFICATIONS

In this section Takenaga's method [9] for evaluating P when a is not small is summarized. Changes were made in his analysis in order to make it efficient for calculation. In addition, it was extended to include the direct computation of Q. These modifications will be given in detail. Basically, the procedure is based on expansions in terms of incomplete normal moment functions, [7], and it works best when a and x are close and large. Even though the method can be used for a as small as 20, it is called in the program only when $a \geqslant 100$, (and $3a \leqslant 4x \leqslant 5a$). Therefore, it will be assumed in discussing the method that $a \geqslant 100$.

Let

$$t = [(z + \alpha)/\beta]^3, \tag{47}$$

where α and β are parameters to be specified. With z as a new integration variable and a replaced by b + 1, (3) transforms to

$$\gamma(b + 1, x) = \int_r^s \frac{3}{\beta} \left(\frac{z + \alpha}{\beta}\right)^{3b+2} \exp\left[-\left(\frac{z + \alpha}{\beta}\right)^3\right] dz , \tag{48}$$

where

$$r = -\alpha , \quad s = \beta x^{1/3} - \alpha , \quad b = a - 1 . \tag{49}$$

We can also write from (1),

$$P(a, x) = P(b + 1, x) = \int_r^s S(b, z)\, dz , \tag{50}$$

where

$$S(b, z) \equiv \left\{ \frac{3}{\beta} \left(\frac{z + \alpha}{\beta}\right)^{3b+2} \exp\left[-\left(\frac{z + \alpha}{\beta}\right)^3\right]\right\} \Big/ \Gamma(b + 1) . \tag{51}$$

Then

$$\log S(b, z) \sim \log \frac{3}{\beta} + (3b + 2) \log \frac{\alpha}{\beta} + (3b + 2) \log (1 + z/\alpha) \tag{52}$$

$$- \left(\frac{\alpha}{\beta}\right)^3 \left[1 + 3 \frac{z}{\alpha} + 3 \left(\frac{z}{\alpha}\right)^2 + \left(\frac{z}{\alpha}\right)^3\right]$$

$$- \left[\left(b + \frac{1}{2}\right) \log b - b + \frac{1}{2} \log 2\pi + \bar{L}(b)\right],$$

where the quantity in brackets of the last line represents the asymptotic expansion of $\log \Gamma(b + 1) = \log b + \log \Gamma(b)$, (see (26)). More specifically,

$$\log \Gamma(b + 1) \sim \left(b + \frac{1}{2}\right) \log b - b + \frac{1}{2} \log 2\pi + \bar{L}(b), \tag{53}$$

with

$$\bar{L}(b) = \frac{1}{12b} - \frac{1}{360b^3} + \frac{1}{1260b^5} - \frac{1}{1680b^7} + \frac{1}{1188b^9} - \cdots, \tag{54}$$

(In (27), we have denoted the first four terms of $\bar{L}(a)$ by $L(a)$). In (52), we make the substitution

$$\log(1 + z/\alpha) = - \sum_{k=1}^{\infty} \frac{1}{k} (-z/\alpha)^k, \quad -1 < z/\alpha < 1. \tag{55}$$

This series can be used for the range of values of (a, x) for which the method is to be applied. Indeed, taking

$$\alpha = 3(b + 2/3)^{1/2} \tag{56}$$

$$\beta = 3(b + 2/3)^{1/6} \tag{57}$$

and noting from (47) that z is an increasing function of t, we have from (49)

$$-1 \leqslant z/\alpha \leqslant (\beta/\alpha)\, x^{1/3} - 1 = [x/(b + 2/3)]^{1/3} - 1 = s/\alpha. \tag{58}$$

In the program, we use (50) only for cases when $x \leq b + 2/3$, (a corresponding expression for $Q(b + 1, x)$ is used when $x > b + 2/3$ and will be discussed later in this section). Consequently, for $x \leq b + 2/3$, (58) shows $z/a \leq 0$ so that the right hand inequality in (55) is satisfied, (it is interesting to note that, regardless of the constraint on x, $z/a < 1$ provided $0 \leq x < 8(b + 2/3)$ which always holds for $a \geq 6$ and $\epsilon \leq P \leq 1 - \epsilon$, where $\epsilon_1 \leq \epsilon$. This follows from the results of the previous section).

The left inequality in (55), on the other hand, cannot be satisfied, because z takes the value $-a$ when $t = 0$. This difficulty is circumvented in the following way. Let

$$P(a, x) = P(a, \underline{X}) + \int_{\underline{X}}^{x} e^{-t}\, t^{b}\, dt/\Gamma(a)\,, \tag{59}$$

where $a = b + 1$, as noted above, and

$$\underline{X} = a\left(1 - \frac{1}{9a} - \tilde{y}/3\sqrt{a}\right)^3, \quad \tilde{y} = 11.0\,. \tag{60}$$

Then by the results of the previous section (see (33) with $\bar{y}$ replaced by $\tilde{y}$) $P(a, \underline{X}) < 4 \times 10^{-28}$. Hence, we can take the second term on the right hand side of (59) in place of $P(a, x)$ with an insignificant loss in relative accuracy. Under these conditions, we get from (47), that when $t = \underline{X}$, $z = \underline{r}$, and

$$\underline{r} = \beta a^{1/3}\left(1 - \frac{1}{9a} - \tilde{y}/3\sqrt{a}\right) - a \tag{61}$$

$$= a\left\{[a/(a - 1/3)]^{1/3}\left(1 - \frac{1}{9a} - \tilde{y}/3\sqrt{a}\right) - 1\right\}$$

$$= a\left\{\left(1 + \frac{1}{9a} + \frac{2}{81}\frac{1}{a^2} + \cdots\right)\left(1 - \frac{1}{9a} - \tilde{y}/3\sqrt{a}\right) - 1\right\}$$

$$> a\left\{\left(1 + \frac{1}{9a} + \frac{2}{81a^2}\right)\left(1 - \frac{1}{9a} - \tilde{y}/3\sqrt{a}\right) - 1\right\} > \frac{a\tilde{y}}{3\sqrt{a}}\left(1 + \frac{9}{a} + \frac{2}{81a^2}\right)$$

Thus, since in our case $a \geq 100$, we see from (61) and (58) that

$$\frac{\tilde{y}}{3\sqrt{a}}\left(1 + \frac{1}{9a} + \frac{2}{81a^2}\right) < \underline{r}/a \leq z/a \leq s/a \leq 0\,.$$

This justifies the use of (55), since for $\tilde{y} = 11.0$ and $a \geq 100$

$$\frac{\tilde{y}}{3\sqrt{a}} \left(1 + \frac{1}{9a} + \frac{2}{81a^2}\right) < 0.37$$

The choices for α and β, given by (56) and (57), were shown by Takenaga, after using (55) in (52), to result in zero coefficients for the terms z/α, $(z/\alpha)^3$, $\log b$, b in (52). Thus, with

$$\log \alpha/\beta = (1/3) \log(b + 2/3) = (1/3) [\log b + \log(1 + 2/3b)] ,$$

he found

$$\log S(b, z) = \log C - \frac{1}{2} \log 2\pi - \frac{z^2}{2} - \frac{\alpha^2}{3} \sum_{k=4}^{\infty} (1/k) (-z/\alpha)^k , \tag{62}$$

where

$$\log C = (b + 1/2) \log(1 + 2/3b) - 2/3 - \bar{L}(b) \tag{63*}$$

$$= -(b + 1/2) \sum_{k=1}^{\infty} (1/k)(-2/3b)^k - 2/3 - \bar{L}(b)$$

$$= \frac{1}{36b} - \frac{1}{81b^2} + \frac{1}{360b^3} + \frac{2}{1215b^4} - \frac{691}{306180b^5} + \frac{16}{15309b^6}$$

$$- \frac{373}{3674160b^7} + \frac{80}{177147b^8} - \frac{44069}{38972340b^9} + \cdots .$$

Then from these results, he obtained

$$S(b, z) = C \left\{ \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \exp\left[-\frac{\alpha^2}{3} \sum_{k=4}^{\infty} (1/k)(-z/\alpha)^k \right] \right\} \tag{64}$$

---

*The coefficients in (63) and (65), or (68) and (80), were computed by Alfred Morris with his "Flap" Algebraic Routine, [5]. Additional coefficients are given in Appendix B. In [9] there is a misprint in (14) and another in (19) of that paper.

Now expanding the second exponential function of (64) in powers of z/a, one gets the power series in (65), namely

$$\exp\left[-\frac{a^2}{3}\sum_{k=4}^{\infty}(1/k)(-z/a)^k\right] = 1 - \frac{1}{12a^2}z^4 - \frac{1}{18a^4}\left(z^6 - \frac{1}{16}z^8\right) \tag{65}$$

$$-\frac{1}{24a^6}\left(z^8 - \frac{37}{225}z^{10} + \frac{1}{432}z^{12}\right)$$

$$\vdots$$

$$-\frac{1}{a^{24}}\left(\frac{1}{78}z^{26} - \cdots - 2.3414\ 67226\ 51062 \times 10^{-22}\ z^{48}\right)$$

$$+\frac{1}{15a^3}z^5 + \frac{1}{3a^5}\left(\frac{z^7}{7} - \frac{z^9}{60}\right) + \frac{1}{27a^7}\left(z^9 - \frac{29}{140}z^{11} + \frac{1}{160}z^{13}\right)$$

$$+\frac{1}{a^9}\left(\frac{1}{33}z^{11} - \frac{193}{22680}z^{13} + \frac{1187}{2268000}z^{15} - \frac{1}{155520}z^{17}\right)$$

$$\vdots$$

$$+\frac{1}{a^{25}}\left(\frac{1}{81}z^{27} - \cdots - 2.2478\ 08537\ 45020 \times 10^{-21}\ z^{49}\right) + \cdots .$$

In order now to carry out the integration of (64), using (65), we need to evaluate integrals of the form

$$\bar{B}_j = \int_{\underline{r}}^{s} z^{2j}\, Z(z)\, dz, \quad \bar{A}_j = \int_{\underline{r}}^{s} z^{2j+1}\, Z(z)\, dz, \tag{66}$$

where j is a non-negative integer (in the program $j \leqslant 24$) and

$$Z(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}.$$

In [9], the indefinite integration of (64), using (65), is carried out, and a cumbersome procedure is suggested for evaluating the constant of integration. The approach below is more direct and easier to apply.

We express the integration of (64) in inverse powers of a,

$$\int_{\underline{r}}^{s} S(b, z)\, dz = C \sum_{k=0}^{K} \bar{d}_k\, a^{-k}, \tag{67}$$

where

$$\bar{d}_0 = \bar{B}_0, \quad \bar{d}_1 = 0, \quad \bar{d}_2 = \frac{1}{12}\bar{B}_2, \quad \bar{d}_3 = \frac{1}{15}\bar{A}_2,$$

and in general for $k \geq 2$,

$$\bar{d}_k = \begin{cases} \displaystyle\sum_{(k+2)/2}^{k} N_{kj}\, \bar{B}_j, & (k \text{ even}, N_{00} = 1) \\ \displaystyle\sum_{(k+1)/2}^{k-1} M_{kj}\, \bar{A}_j, & (k \text{ odd}). \end{cases} \tag{68}$$

The constants N_{kj}, M_{kj} which result from the expansion in (65) are given in Appendix B. Some particular values are $N_{22} = \frac{1}{12}$, $M_{7,5} = -29/(27 \times 140)$. If k is even, a typical term of the sum on the right hand side of (68), multiplied by a^{-k}, is given by

$$a^{-k} N_{kj}\, \bar{B}_j = N_{kj}\, a^{-k} \int_{\infty}^{s} z^{2j} Z\, dz - N_{kj}\, a^{-k} \int_{\infty}^{\underline{r}} z^{2j} Z\, dz, \tag{69}$$

or in the case of k odd,

$$a^{-k} M_{kj}\, \bar{A}_j = M_{kj}\, a^{-k} \int_{\infty}^{s} z^{2j+1} Z\, dz - M_{kj}\, a^{-k} \int_{\infty}^{\underline{r}} z^{2j+1} Z\, dz. \tag{70}$$

We want to show that the last term on the right hand side of these identities can be dropped for $b (= a - 1) \geq 99$ with no significant loss in relative accuracy. First, we obtain an upper bound on $\underline{r}$. From (61),

$$\underline{r} = a \left\{ \left(\frac{b+1}{b+2/3}\right)^{1/3} \left(1 - \frac{1}{9(b+1)}\right) - 1 - \tilde{y} \left(\frac{b+1}{b+2/3}\right)^{1/3} \Big/ 3\sqrt{b+1} \right\}.$$

Then

$$\left(\frac{b+1}{b+2/3}\right)^{1/3}\left|1-\frac{1}{9(b+1)}\right| = \left|1+\frac{1}{3(b+2/3)}\right|^{1/3}\left|1-\frac{1}{9(b+2/3)[1+1/3(b+2/3)]}\right|$$

$$= \left(1+\frac{3}{a^2}\right)^{1/3}\left|1-\frac{1}{a^2(1+3/a^2)}\right| \tag{71}$$

$$< \left(1+\frac{1}{a^2}-\frac{1}{a^4}+\frac{5}{3a^6}\right)\left(1-\frac{1}{a^2}+\frac{3}{a^4}-\frac{9}{a^6}+\frac{27}{a^8}\right)$$

$$< 1+\frac{1}{a^4}-\frac{10}{3}\frac{1}{a^6}+\frac{40}{3}\frac{1}{a^8} < 1+1/a^4, \quad (a \geqslant 100).$$

Therefore,

$$\underline{r} < -\tilde{y}\left(\frac{b+2/3}{b+1}\right)^{1/6}+\frac{1}{a^3} < -10.9938 \quad (b \geqslant 99, \quad \tilde{y} = 11.0).$$

Now, letting $v = 10.9938$ and taking $k = j = 0$ in (69), we have

$$\int_{-\infty}^{\underline{r}} Z\,dz = \int_{\underline{r}}^{\infty} Z\,dz < \int_{v}^{\infty} Z\,dz < 2.05 \times 10^{-28} \quad \text{(See [1, p. 972])}$$

Consequently, the last term in (69), with $k = j = 0$, can be neglected, (the constant C, in (67), does not affect this conclusion, since it is always close to one, for $a \geqslant 100$, as (18) or (63) shows).

Now consider the absolute value of a particular term in (67) for $k > 0$ and even.

$$|N_{kj}|\, a^{-k}\int_{-\infty}^{\underline{r}} z^{2j}\, Z\,dz < |N_{kj}|\, a^{-k}\int_{v}^{\infty} z^{2j}\, Z\,dz.$$

Then with $u = z^2/2$ and the use of (45),

$$|N_{kj}|\, a^{-k} \int_v^{\infty} z^{2j}\, Z\, dz = |N_{kj}| \frac{a^{-k}\, 2^j}{2\sqrt{\pi}} \int_{v^2/2}^{\infty} u^{j-1/2}\, e^{-u}\, du \tag{72}$$

$$= |N_{kj}| \frac{a^{-k}\, 2^j}{2\sqrt{\pi}}\, Q(j + 1/2,\ v^2/2)\, \Gamma(j + 1/2)$$

$$< \frac{|N_{kj}|\, a^{-k}\, 2^j}{2\sqrt{\pi}} \; \frac{e^{-v^2/2}\, (v^2/2)^{j+1/2}}{[(v^2/2)+1-j-1/2]} \qquad \text{(See (45))}$$

$$= \frac{|N_{kj}|\, a^{-k}}{\sqrt{2}\,\sqrt{\pi}} \; \frac{e^{-v^2/2}\, v^{2j+1}}{[v^2+1-2j]}$$

A similar analysis for k odd shows

$$|M_{kj}|\, a^{-k} \int_v^{\infty} z^{2j+1}\, Z\, dz \leqslant |M_{kj}| \frac{a^{-k}}{\sqrt{2\pi}} \; \frac{e^{-v^2/2}\, v^{2j+2}}{(v^2 - 2j)} \tag{73}$$

It can be shown for every case considered in the program, i.e., $0 \leqslant k \leqslant 24$, $(k + 2)/2 \leqslant j \leqslant k$ (k even), $(k + 1)/2 \leqslant j \leqslant (k - 1)$ (k odd) that by bounding the right hand sides of (72) and (73), the left hand sides of these inequalities are very small. For example setting $b = 99$ (for which a^{-k} takes its largest value), $k = 23$, $j = 22$, $M_{23,22} \cong 3 \times 10^{-19}$ (see Appendix B), we have from (73)

$$M_{23,22}\, a^{-23} \int_v^{\infty} z^{45}\, Z\, dz < \frac{3 \times 10^{-19} \times 1.1 \times 10^{-34}}{\sqrt{2\pi} \times 76.86} \times 4.5\ 10^{21} = O(10^{-34}).$$

For the same value of k with $j = 12$, $M_{23,12} \cong 0.0133 \cdots$,

$$M_{23,12}\, a^{-23} \int_v^{\infty} z^{25}\, Z\, dz < \frac{1.4 \times 10^{-2} \times 1.1 \times 10^{-34}}{\sqrt{2\pi} \times 95.86} \times 6.7 = O(10^{-37}).$$

Thus, the second terms on the right hand sides of (69) and (70) will be discarded, or equivalently the integrals $\bar{B}_j$ and $\bar{A}_j$ in (66) are replaced by

$$B_j = \int_{-\infty}^{v} z^{2j}\, Z\, dz \tag{74}$$

$$A_j = \int_{-\infty}^{v} z^{2j+1}\, Z\, dz \tag{75}$$

These integrals are easily evaluated on a computer by using the recurrence relations

$$B_j = (2j - 1)\, B_{j-1} - s^{2j-1}\, Z(s) \tag{76}$$

$$A_j = 2j\, A_{j-1} - s^{2j}\, Z(s)\ , \tag{77}$$

with

$$B_0 = \int_{-\infty}^{s} Z\, dz = \frac{1}{2}\, \mathrm{erfc}\, (|s|/\sqrt{2}\,), \qquad s \leqslant 0 \tag{78}$$

$$A_0 = -\frac{1}{\sqrt{2\pi}}\, e^{-s^2/2} = -Z(s)\ . \tag{79}$$

There is no problem to compute B_0 since a very efficient and accurate error function routine is available. It is based on Cody's minimax approximations, [2], and it yields 14 significant digits for erf or erfc depending on the magnitude of the argument. For completeness, the particular Cody approximations that we use are specified in Appendix A. A discussion of how s is computed is given on page 41.

Finally the quantity C, whose logarithm is given by (63), is expressed as a power series in inverse powers of b in order to evade computing log C from the series in (63), and subsequently C from exp (log C). Thus we obtain

$$C = 1 + \frac{1}{36}\, b^{-1} - \frac{31}{2592}\, b^{-2} + \frac{3413}{1399680}\, b^{-3} + \frac{361733}{201553920}\, b^{-4} \tag{80*}$$

$$- \frac{113888281}{50791587840}\, b^{-5} + \frac{7565202533}{7836416409600}\, b^{-6} - \cdots .$$

Takenaga's procedure is extended to advantage by using the same approach when $x > b + 2/3 = a - 1/3$ to compute Q instead of P. As noted earlier, P in this case can be close to one so that Q must be computed directly to maintain the specified relative accuracy.

*See footnote on page 25.

The computation of Q only requires changing the limits of integration in (74) and (75) from $-\infty$ and s to s and ∞, respectively. Thus (76) – (79) are replaced by

$$B_j = (2j - 1)\, B_{j-1} + s^{2j-1}\, Z(s) \tag{81}$$

$$A_j = 2j\, A_{j-1} + s^{2j}\, Z(s) \tag{82}$$

$$B_0 = \frac{1}{\sqrt{2\pi}} \int_s^{\infty} e^{-t^2/2}\, dt = \frac{1}{2}\, \mathrm{erfc}\, (s/\sqrt{2}) \tag{83}$$

$$A_0 = \frac{1}{\sqrt{2\pi}}\, e^{-s^2/2} = Z(s) \tag{84}$$

Certainly these changes are quite simple. We also note from (49) or (58) that $s > 0$ since $x > b + 2/3$.

In order to establish these results for $Q(b + 1, x)$, we require the integration of $S(b, z)$ in (51) from s to ∞, with $s > 0$. Again this raises the question, when we proceed as before, concerning the validity of using the expansion in (55). In order to justify its use we must show that $|z/a| < 1$ for all z in the domain of integration. Clearly this will not be the case if $z \in [s, \infty]$. We get around this problem much like before. Note that since $x > b + 2/3$, a lower bound on z/a is easily found, namely

$$(z/a) \geqslant s/a = [x/(b + 2/3)]^{1/3} - 1 > 0$$

An upper bound on z/a is found by considering

$$Q(a, x) = Q(a, \overline{X}) + \int_x^{\overline{X}} e^{-t}\, t^b\, dt\,, \quad a = b + 1, \tag{85}$$

where $\overline{X} = a(1 - \frac{1}{9a} + \tilde{y}/3\sqrt{a}\,)^3 > x > 0$. It follows from (46) and [1, p. 972] that for $\tilde{y} = 11.0$, $Q(a, \overline{X}) < 7 \times 10^{-29}$. Thus, we can replace $Q(a, x)$ by the second integral on the right hand side of (85). We get an upper bound on z/a by using $t \leqslant \overline{X}$ in (47). Indeed,

$$z \leqslant a\,\left[(\beta/a)\,\overline{X}^{1/3} - 1\right] = a \left\{ \left(\frac{b+1}{b+2/3}\right)^{1/3} \left(1 - \frac{1}{9a} + \tilde{y}/3\sqrt{a}\right) - 1 \right\}$$

$$< \left(\frac{b + 2/3}{b + 1}\right)^{1/6} \tilde{y} + 1/a^3 < 10.9939 = v, \quad \text{(See (71))}.$$

Since $b \geqslant 99$, we have, by noticing that v/a is a decreasing function of b,

$$0 < s/a \leqslant z/a < v/a < 0.37.$$

This result allows us to use (55) when $s > 0$, $(x > b + 2/3)$.

The algebraic procedures which yield (65) in inverse powers of a, and the subsequent integration show that individual terms contain integrals of the form (after dropping $Q(a, \bar{X})$ in (85))

$$\bar{B}_j = \int_s^v z^{2j}\, Z\, dz, \qquad A_j = \int_s^v z^{2j+1}\, Z\, dz, \tag{86}$$

where

$$Z = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}.$$

Proceeding as in the case for $x \leqslant b + 2/3$, the expressions corresponding to (69) and (70) are given by

$$a^{-k} N_{kj}\, \bar{B}_j = N_{kj}\, a^{-k} \int_s^{\infty} z^{2j}\, Z\, dz - N_{kj}\, a^{-k} \int_v^{\infty} z^{2j}\, Z\, dz \tag{87}$$

$$a^{-k} M_{kj}\, \bar{A}_j = M_{kj}\, a^{-k} \int_s^{\infty} z^{2j+1}\, Z\, dz - M_{kj}\, a^{-k} \int_v^{\infty} z^{2j+1}\, Z\, dz. \tag{88}$$

We note that the coefficients of (68), N_{kj} and M_{kj}, are unchanged and that (68) holds with $\bar{B}_j$ and $\bar{A}_j$ given by (86). But, we have argued above (see (72), (73)) that the second term on the right of (87) and (88) can be dropped or, equivalently, the integrals $\bar{B}_j$ and $\bar{A}_j$ in (86) can be replaced by

$$B_j = \int_s^{\infty} z^{2j}\, Z\, dz \qquad A_j = \int_s^{\infty} z^{2j+1}\, Z\, dz. \tag{89}$$

It is interesting to note that these integrals were obtained directly by formally integrating (64) from s to ∞. They are evaluated by the recurrence relations (81) and (82).

The final integrated expressions for $P(b + 1, x)$ and $Q(b + 1, x)$ are given by (17) and again in (115) of the next section where the computer program is discussed.

5. DESCRIPTION OF PROGRAM

In this section we summarize the equations and inequalities on which the CDC-6700 computing program is based. In addition, the general features of the program are described and a master flowchart is included at the end of this section as Figure 3.

The program requires as input 5 quantities, the two locations where two arguments a and x are stored, the two output locations for P and Q, and a location which is used to specify whether $\epsilon_1 (= 5 \times 10^{-13})$, $\epsilon_2 (= 5 \times 10^{-7})$, or $\epsilon_3 (= 5 \times 10^{-4})$ is to be used.

If ϵ_1 (ϵ_2) $[\epsilon_3]$ is specified, then the output quantities P and Q are given correctly to within one unit in the twelfth (sixth) [third] significant digit. Of course, if (a, x) is outside the "zone of computation" P and Q may have no correct significant digits since they are simply given as 1 and 0 or 0 and 1. The absolute error in such cases, of course, will always be within the specified ϵ.

In order to give some estimate of the average computing time per case, two grids of points in the ax-plane were used for $\epsilon = \epsilon_1$ and two others for $\epsilon = \epsilon_2$. The two grids for ϵ_1 are specified first, identifying them as A_1 and B_1.

Grid A_1	$\epsilon = \epsilon_1$	Grid B_1
a = 0.0001 (0.001) 0.05 x = 0 (0.1) 126.1		a = 0.0001 (0.001) 0.05 x = 0 (0.1) X (a)
a = 0.1 (0.1) 31 x = 0 (0.5) 1260		a = 0.1 (0.1) 31 x = 0 (0.5) X (a)
a = 31.0 (0.5) 99.5 x = 0 (0.5) 1260		a = 31.0 (0.5) 99.5 x = $\underline{x}_1$ (0.5) X (a)
a = 100.1 (5) 1000.1 x = 0 (10) 1260		a = 100.1 (5) 1000.1 x = $\underline{x}_1$ (10) X (a)

X (a) = minimum grid value of x, given a, such that $Q \leq \epsilon$.

$\underline{x}_1$ = value of x, given a, for which $P(a, x) \sim \epsilon$. See (33).

$$\underline{x}_1 = a\left(1 - \frac{1}{9a} - \bar{y}/3\sqrt{a}\right)^3, \quad \bar{y} = 7.1306, \quad (\bar{y} = 4.892, \epsilon = \epsilon_2).$$

All the grid points, (a, x), specified are considered.

Denoting the corresponding grids for $\epsilon = \epsilon_2$ by A_2 and B_2, we have

Grid A_2	$\epsilon = \epsilon_2$	Grid B_2
a = 0.0001 (0.001) 0.05 x = 0 (0.1) 117.1		a = 0.0001 (0.001) 0.05 x = 0 (0.1) X (a)
a = 0.1 (0.1) 20 x = 0 (0.5) 1170		a = 0.1 (0.1) 20 x = 0 (0.5) X (a)
a = 20.5 (0.5) 99.5 x = 0 (0.5) 1170		a = 20.5 (0.5) 99.5 x = $\underline{x}_1$ (0.5) X (a)
a = 100.1 (5) 1000.1 x = 0 (10) 1170		a = 100.1 (5) 1000.1 x = $\underline{x}_1$ (10) X (a)

The major portion of the points, (a, x), in the A grids lie outside the "zone of computation," whereas almost all of the points in the B grids lie within the "zone of computation." The A grids were designed to get a representative average time per case in the situation where any point in the region $0.0001 \leqslant a \leqslant 1000.1$ $0 \leqslant x \leqslant U^*$ (where U denotes the maximum value on x) is equally likely to occur as input. On the other hand, the B grids were set up to obtain a represen: ative average time per case in those situations where the input point (a, x) is most likely to occur in the "zone of computation." In Table 3, the results for the various grids are summarized.

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*We note for a = 0.0001 (0.001) 0.05 U is taken as 126.1 for $\epsilon = \epsilon_1$ and 117.1 for $\epsilon = \epsilon_2$.

Table 3. Summary of Grid Results

EQ. / GRID	(10)	(12)	(13)	(15)	(16)	(17)	E	TOTAL CASES	AVG. TIME (msec)
GRID A_1	20021	39089	1891	17520	769	5872	1131595	1216757	0.75
GRID B_1	19958	39155	1879	17512	769	5860	2274	87407	2.02
GRID A_2	12920	20708	561	5606	769	4020	876965	921549	0.68
GRID B_2	12788	20783	551	5598	769	3993	1687	46169	1.59

The numbers in parentheses refer to the equations in the text, e.g., (10) was used 20021 times in Grid A_1. The column headed E gives the number of cases that were recognized by the program to be outside the "zone of computation." The average time per case is given in the last column in milliseconds. The much smaller computing times for the A grids are to be expected, since most of the cases were outside the "zone of computation."

The flow of the program, depending on the input, should be easy to follow from the flow chart at the end of this section. The circled numbers are used to denote subprograms for computing P or else Q, and the number itself corresponds to the equation in Section 2 upon which the subprogram is based. For example, 10 refers to that subprogram which uses (10) for the computation of P; 12 refers to that subprogram which uses (12) to determine Q. The quantities $\overline{F}$ and $\underline{F}$ are defined by (34) and (35). The term $R(a, x) \equiv e^{-x} x^a/\Gamma(a)$, (see also (11)), and ϵ refers to either ϵ_1, ϵ_2, or ϵ_3, (see (6) and (7)).

We continue with a brief description of the algorithm used with each subprogram. It is assumed (a, x) is in the "zone of computation" and that all the subprograms except 17 require $a < 100$. *

If $a \geqslant 1$, then 10 is used if $a \geqslant x$. Also, 10 is used when 2a is not an integer provided $1 < a < x \leqslant \ell n\ 10$ or $a < 1$, $x < 1.5$ and $P \leqslant 0.90$, or $a < 1$, $x \geqslant 1.5$ and

*If $a \geqslant 100$, 10 is used if $3a > 4x$ and 12 is used if $4x > 5a$.

$R > 0.101\ x\ (x + 2 - a)/(x + 1)$. Let s_{n-1} denote the $(n - 1)$ st term of the series in (10), then the n^{th} term is found from

$$s_n = \left(\frac{x}{a + n}\right) s_{n-1}, \quad s_0 = 1, \quad n = 1, 2, \cdots \tag{90}$$

The summation of the series is terminated at $n = N$ when

$$s_N \leqslant \epsilon/2 \ \text{ and } \ x/(a + N + 1) \leqslant 2/3,$$

which assures a truncation error, $\sum_{N+1}^{\infty} s_k < \epsilon$, (see Appendix D).

Program (12) applies when $1 < a < x$, and $x \geqslant 31$ if $\epsilon = \epsilon_1$; $x \geqslant 17$ if $\epsilon = \epsilon_2$; $x \geqslant 9.7$ if $\epsilon = \epsilon_3$. The $(n + 1)$ st term of the asymptotic series (12) for Q is given by

$$V_{n+1} = \left(\frac{a - 1 - n}{x}\right) V_n, \quad V_0 = 1, \quad n = 0, 1, \cdots. \tag{91}$$

The summation is terminated when

$$|V_{n+1}| < \epsilon. \tag{92}$$

Program (13) is used when $a \leqslant x < 31$ for $\epsilon = \epsilon_1$, $a \leqslant x < 17$ for $\epsilon = \epsilon_2$, or $a \leqslant x < 9.7$ for $\epsilon = \epsilon_3$, and if 2a is an integer. The algorithm for computing Q from (13) then is given by

$$W(k + g - 1, x) = \left(\frac{x}{k + g - 1}\right) W(k + g - 2, x) \tag{93}$$

$$Q(k + g, x) = Q(k + g - 1, x) + W(k + g - 1, x), \quad k = 1, 2, \cdots, a - g, \tag{94}$$

where $g = 1$ if a is an integer or $g = 1/2$ if a is not an integer. Also

$$W(k + g - 1, x) = R(k + g - 1, x)/(k + g - 1) \tag{95}$$

$$W(g - 1, x) \equiv \begin{cases} (\pi x)^{-1/2} e^{-x} & g = 1/2 \\ e^{-x} & g = 1 \end{cases} \tag{96}$$

$$Q(g, x) = \begin{cases} \operatorname{erfc}(\sqrt{x}) & g = 1/2 \\ e^{-x} & g = 1 \quad \text{(See (14))}. \end{cases} \tag{97}$$

Program (15) is based on a continued fraction expansion for Q. It is used when 2a is not an integer and $1 < a < x$ with $\ell n\, 10 < x < 31$ for $\epsilon = \epsilon_1$ ($\ell n\, 10 < x < 17$ for $\epsilon = \epsilon_2$, or $\ell n\, 10 < x < 9.7$ for $\epsilon = \epsilon_3$) or $a < 1$, $x \geq 1.5$ and $R \leq 0.101\, x(2 + x - a)/(x + 1)$. The algorithm for computing two successive approximates, based on (15), in each iteration follows. Let

$$D_1 = D_2 = 1, \quad E_1 = x, \quad E_2 = (x + 1 - a), \quad n = 1, 2, \cdots .$$

Then

$$\begin{cases} D_{2n+1} = x\, D_{2n} + n\, D_{2n-1} \\ E_{2n+1} = x\, E_{2n} + n\, E_{2n-1} \end{cases} \tag{98}$$

Increase n to n + 1

$$\begin{cases} D_{2n} = D_{2n-1} + (n - a)\, D_{2n-2} \\ E_{2n} = E_{2n-1} + (n - a)\, E_{2n-2} . \end{cases} \tag{99}$$

The procedure is stopped when

$$\left| \frac{D_{2n}}{E_{2n}} - \frac{D_{2n-1}}{E_{2n-1}} \right| < \epsilon \left| \frac{D_{2n}}{E_{2n}} \right| . \tag{100}$$

Program (16) is used when $a < 1$, $x < 1.5$ and $P > 0.90$. From (16), three series are evaluated, J, L, H.

We have

$$J = -a \sum_{k=1}^{\infty} \frac{(-x)^k}{(a + k)\, k!} = -a \sum_{1}^{\infty} J_k = -a \sum_{1}^{\infty} T_k/(a + k) , \tag{101}$$

then

$$T_k = (-x/k)\, T_{k-1}, \quad J_k = T_k/(a + k) \quad k = 2, \cdots , \tag{102}$$

$$T_1 = -x, \quad J_1 = -\frac{x}{a + 1} . \tag{103}$$

The iterations are stopped when

$$|J_k| < -\epsilon \sum_{i=1}^{k} J_i. \tag{104}$$

For L, we have

$$L = a \ln x \left[1 + \sum_{k=1}^{\infty} (a \ln x)^k/(k+1)!\right] = a \ln x \left[1 + \sum_{k=1}^{\infty} L_k\right]. \tag{105}$$

Then

$$L_k = \left(\frac{a \ln x}{k+1}\right) L_{k-1}, \quad L_1 = (a \ln x)/2, \quad k = 2, \cdots. \tag{106}$$

These iterations are terminated when

$$|L_k| \leq \epsilon/2 \tag{107}$$

The H series,

$$H = \begin{cases} \sum_{2}^{17} C_k a^{k-1} & 0 < a < 1/2 \\ \\ (1/a) \left[(1-a) + \sum_{2}^{17} C_k (a-1)^{k-1}\right] & 1/2 \leq a < 1, \end{cases} \tag{108}$$

is terminated when $k = 17$ or before if $|K_n| < (\epsilon/2) \left|\sum_{2}^{n} K_k\right|$, where $K_k = C_k a^{k-1}$ if $0 < a < 1/2$ or $K_k = C_k(a-1)^{k-1}$ if $1/2 \leq a < 1$. We note also that the H-series make up part of the series that are used for evaluating $1/\Gamma(\lambda)$ when $0 < \lambda < 1$, (see (25)). Consequently, when $0 < a < 1$, H is obtained during the computation of $1/\Gamma(a)$ and stored. The evaluation of $1/\Gamma(a)$ is taken up later in this section on page 42 and in Appendix A.

Program 17 is the only one used when $a \geqslant 100$, and $3a \leqslant 4x \leqslant 5a$. The individual integrals A_k, B_k that appear in (17) are given by (20) and (21) if $x \leqslant a - 1/3$ and by (23) and (24) if $x > a - 1/3$. In case the first inequality is satisfied, P is computed, and the recurrence relations for evaluating the integrals of (21) and (20) are given by (76) and (77). In case the second inequality holds, Q is computed, and the recurrence relations for evaluating (24) and (23) are given by (81) and (82).

We write again the basic equations for 17, with (76), (77), (81), (82) slightly changed for greater efficiency in computation. Let

$$B_j = Z(s)\, b_j \qquad A_j = Z(s)\, a_j \qquad \left(Z(s) = \frac{1}{\sqrt{2\pi}}\, e^{-s^2/2}\right). \tag{109}$$

Then

$$\begin{cases} b_j = (2j-1)\, b_{j-1} - s^{2j-1} \\ a_j = 2j\, a_{j-1} - s^{2j} \end{cases} \tag{110}$$

replace (76) and (77) and are used when $x \leqslant a - 1/3$ with

$$\begin{cases} b_0 = [1/2Z(s)]\ \mathrm{erfc}\,(|s|/\sqrt{2}) \\ a_0 = -1 . \end{cases} \tag{111}$$

If, on the other hand, $x > a - 1/3$, then using (109), (81) and (82) become

$$\begin{cases} b_j = (2j-1)\, b_{j-1} + s^{2j-1} \\ a_j = 2j\, a_{j-1} + s^{2j} \end{cases} \tag{112}$$

with

$$\begin{cases} b_0 = [1/2Z(s)]\ \mathrm{erfc}\,(s/\sqrt{2}) \\ a_0 = 1 . \end{cases} \tag{113}$$

In all cases

$$s = 3(a - 1/3)^{1/2} \left\{ [x/(a - 1/3)]^{1/3} - 1 \right\}, \quad \text{(See (22))}. \tag{114}$$

Thus, in terms of a_k, b_k, instead of A_k, B_k, (17) becomes

$$T(a, x) = Z(s)\, C \left[b_0 - \frac{1}{12a^2} b_2 \quad \frac{1}{18a^4} \left(b_3 \quad \frac{1}{16} b_4 \right) \right. \tag{115}$$

$$- \frac{1}{24a^6} \left(b_4 - \frac{37}{225} b_5 + \frac{1}{432} b_6 \right)$$

$$- \frac{1}{30a^8} \left(b_5 - \frac{743}{3024} b_6 + \frac{49}{4320} b_7 \quad \frac{5}{82944} b_8 \right)$$

$$\vdots$$

$$\frac{1}{a^{24}} \left(\frac{1}{78} b_{13} - \cdots - 2.3414\,67226\,51062 \times 10^{-22}\, b_{24} \right)$$

$$+ \frac{a_2}{15a^3} + \frac{1}{3a^5} \left(\frac{a_3}{7} \quad \frac{a_4}{60} \right) + \frac{1}{27a^7} \left(a_4 \quad \frac{29}{140} a_5 + \frac{1}{160} a_6 \right)$$

$$+ \frac{1}{a^9} \left(\frac{a_5}{33} \quad \frac{193}{22680} a_6 + \frac{1187}{2268000} a_7 \quad \frac{1}{155520} a_8 \right)$$

$$\vdots$$

$$\left. + \frac{1}{a^{25}} \left(\frac{a_{13}}{81} - \cdots - 2.2478\,08537\,45020 \times 10^{-21}\, a_{24} \right) \right]$$

where, with $b = a \quad 1$,

$$C = 1 + \frac{1}{36} b^{\;1} \quad \frac{31}{2592} b^{\;2} + \frac{3413}{1399680} b^{\;3} + \frac{361733}{201553920} b^{\;4} \tag{116}$$

$$\frac{113888281}{50791587840} b^{-5} + \frac{7565202533}{7836416409600} b^{\;6} \quad \cdots \quad \text{(See (18))}.$$

The program uses no more terms than are shown for (115) and (116). The numerical coefficients in these equations, and some additional ones for (116), not used in the program, are given in Appendix B.

The right hand side of (115) is computed by the following algorithm:

(a) Compute a_0, b_0, ζ_0 ($= b_0$) (use (111) for $s \leq 0$ or (113) for $s > 0$)

(b) Compute $1/\alpha^2$, $1/\alpha$ ($\alpha = 3(a - 1/3)^{1/2}$)

(c) Set $i = 2$, $\Sigma = \zeta_0$ (Σ denotes accumulated sum)

(d) Compute b_{i-1}, b_i, a_{i-1}, a_i (by (110) for $s \leq 0$ or (112) for $s > 0$)

(e) Compute $\zeta_i \equiv \left(\sum_{1+i/2}^{i} N_{ij}\, b_j \right) \Big/ \alpha^i$, $\zeta_{i+1} = \left(\sum_{1+i/2}^{i} M_{i+1,j}\, a_j \right) \Big/ \alpha^{i+1}$

(See (67) and (68).)

(f) Add ζ_i and ζ_{i+1} to accumulated sum, Σ, to obtain

$$\Sigma = \sum_{k=0}^{i-1} \zeta_k + \zeta_i + \zeta_{i+1} \qquad (\zeta_1 = 0)$$

(g) Is $|\zeta_i| \leq \epsilon\, b_0$, and $|\zeta_{i+1}| \leq \epsilon\, b_0$, or is $i = 24$?

(h) If no in both cases then increase i by 2 and return to (d)

(i) If yes in either case, for (g), proceed to (j)

(j) Compute $Z(s)\, C\Sigma$ and exit.

The calculation of s does not result in any significant increase in relative error in computing P or Q. A simple analysis shows the relative error in P or Q is not greater than twice the absolute error in s. Nevertheless, in order to minimize the absolute error in s, it is evaluated by the algebraic equivalent of (114), namely

$$s = \frac{\alpha\,\delta}{(a - 1/3)} \Bigg/ \left\{ 1 + \left(\frac{x}{a - 1/3} \right)^{1/3} + \left(\frac{x}{a - 1/3} \right)^{2/3} \right\},$$

where

$$\delta = (x - a) + 1/3.$$

If (10), (12), (15), or (16) is used the function R(a, x), as defined in (11) is computed. If $a > 30$ its computation is carried out by using (28) with L(a) given by (27). This eliminates any scaling problem with evaluating R(a, x) for $30 < a$.

If $a \leqslant 30$, then R is evaluated by

$$R(a, x) = [e^{-x+a \ln x}]/\Gamma(a). \tag{117}$$

where the following procedure is used to compute $\Gamma(a)$. Let λ be the fractional part of a.

If $\lambda = 0$, then

$$\begin{cases} \Gamma(j+1) = j\,\Gamma(j), & j = 1, 2, \cdots, a-1, \quad a > 1. \\ \Gamma(1) = 1. & a = 1. \end{cases} \tag{118}$$

If $\lambda \neq 0$, then $1/\Gamma(\lambda)$ is found to 14 significant digits by the polynomials

$$1/\Gamma(\lambda) = \begin{cases} \lambda \left[1 + \sum_{2}^{17} C_k \lambda^{k-1}\right] & 0 < \lambda < 1/2 \\ \\ 1 + \sum_{2}^{17} C_k (\lambda - 1)^{k-1} & 1/2 \leqslant \lambda < 1; \end{cases} \tag{119}$$

the C_k's are given in Appendix A with the rule for terminating these series. Then $\Gamma(a)$ is computed from

$$\begin{cases} \Gamma(j+\lambda) = (j-1+\lambda)\,\Gamma(j-1+\lambda), & j = 1, \cdots, a-\lambda, \quad a > 1 \\ \Gamma(a) = \Gamma(\lambda), & a < 1. \end{cases} \tag{120}$$

For values of $a > 20$ it was observed that the asymptotic series for $\ln \Gamma(a)$, (26), yielded a faster algorithm, in spite of evaluating $\Gamma(a)$ from $e^{\ln \Gamma(a)}$, than the one based on (118) – (120). However, the relative error was larger. In the procedure above 1.5 more significant figures were obtained in some cases, for $\epsilon = \epsilon_1$, than by using the asymptotic series.

The error function, erf(x), and its complement erfc(x) are evaluated by Cody's minimax approximations, [2], (see Appendix A). We note that the error function, or its complement, is needed in (11), (111), (113).

This completes the discussion of the algorithms for the computer program. A few final comments are in order regarding the checkout of the program.

The program was extensively checked by computing P (or Q) for a large range of arguments, (a, x), such that both (10) and (12) could be used for the same arguments. This overlapping procedure was also used with (10) and (13), (10) and (17), (10) and (15), (13) and (15), (12) and (15), (12) and (17). The subprogram (16) was validated by comparing its results with those from a double precision version of (10). A final check that was made compared the computation of Q(x + 1, x) from (17), ($x \geq 100$), with an asymptotic expansion for this quantity given by

$$Q(x+1,x) \simeq \frac{1}{2} + \frac{1}{3}\sqrt{\frac{2}{\pi x}}\left[1 - \frac{23}{180\,x} + \frac{23}{2016\,x^2} + \cdots\right], \quad (x \to \infty). \tag{121}$$

Values of x as large as 10^6 were tested and the computed values of Q by (17) and (121) met the desired relative accuracy for $\epsilon = \epsilon_1$, $\epsilon = \epsilon_2$ and $\epsilon = \epsilon_3$, i.e., correct to within one unit in the 12^{th}, 6^{th} and 3^{rd} significant digit, respectively. For completeness, the method by which (121) was obtained and its leading sixteen terms are given in Appendix C.

The program, as noted earlier, senses if for a given (a, x) P or Q is greater than the specified $\epsilon(\epsilon_1$ or ϵ_2 or $\epsilon_3)$, i.e., if (a, x) is in the zone of computation. If the user desires P or Q computed for values smaller than ϵ, i.e., when (a, x) is outside the zone of computation, this can easily be accomplished by changing one location (ACO) in the program, (see page E-1).

A FORTRAN listing for the entire program is given in Appendix E.

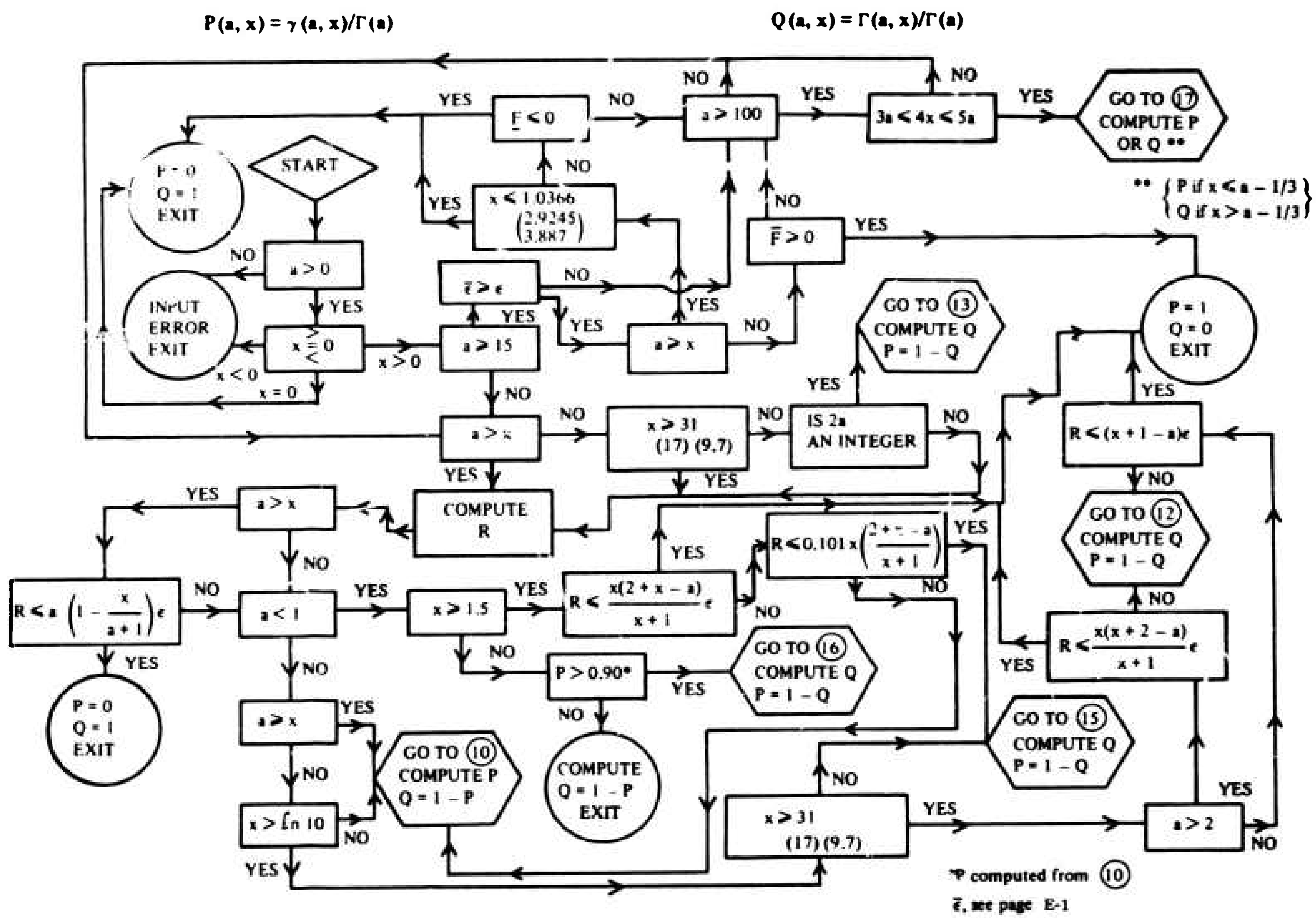

Figure 3. Master Flow Chart $a > 0$ $x > 0$

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APPENDIX A

COMPUTATION OF THE ERROR FUNCTION, AND THE RECIPROCAL OF THE GAMMA FUNCTION OF a WHEN a ≤ 1

COMPUTATION OF erf(x) AND $1/\Gamma(a)$, $a \leq 1$

The formulations for the computation of erf(x) or erfc(x), depending on x, are taken from [2]. They are based on rational Chebyshev approximations.

If $|x| < 0.5$

$$\mathrm{erf}(x) = x \sum_{k=0}^{3} p_k\, x^{2k} \Big/ \sum_{k=0}^{3} q_k\, x^{2k} \qquad \text{(A-1)}$$

$p_0 =$ 2.42667 95523 05318 (2)	$q_0 =$ 2.15058 87586 98612 (2)
$p_1 =$ 2.19792 61618 29415 (1)	$q_1 =$ 9.11649 05404 51490 (1)
$p_2 =$ 6.99638 34886 19136 (0)	$q_2 =$ 1.50827 97630 40779 (1)
$p_3 =$ 3.56098 43701 81539 (−2)	$q_3 =$ 1.0 (0)

The maximum relative error $< 10^{-14.65}$.

If $\frac{1}{2} \leq x \leq 4.0$

$$1 - \mathrm{erf}(x) = \mathrm{erfc}(x) = e^{-x^2} \sum_{k=0}^{7} p_k\, x^{k} \Big/ \sum_{k=0}^{7} q_k\, x^{k} \qquad \text{(A-2)}$$

$p_0 =$ 3.00459 26102 01616 005 (2)	$q_0 =$ 3.00459 26095 69832 933 (2)
$p_1 =$ 4.51918 95371 18729 422 (2)	$q_1 =$ 7.90950 92532 78980 272 (2)
$p_2 =$ 3.39320 81673 43436 870 (2)	$q_2 =$ 9.31354 09485 06096 211 (2)
$p_3 =$ 1.52989 28504 69404 039 (2)	$q_3 =$ 6.38980 26446 56311 665 (2)
$p_4 =$ 4.31622 27222 05673 530 (1)	$q_4 =$ 2.77585 44474 39876 434 (2)
$p_5 =$ 7.21175 82508 83093 659 (0)	$q_5 =$ 7.70001 52935 22947 295 (1)
$p_6 =$ 5.64195 51747 89739 711 (−1)	$q_6 =$ 1.27827 27319 62942 351 (1)
$p_7 =$ −1.36864 85738 27167 067 (−7)	$q_7 =$ 1.00000 00000 00000 000 (0)

The maximum relative error $< 10^{-16.13}$.

If $x \geqslant 4.0$

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x}\left[\frac{1}{\sqrt{\pi}} + \frac{1}{x^2}\sum_{k=0}^{4} p_k\, x^{-2k} \Big/ \sum_{k=0}^{4} q_k\, x^{-2k}\right] \tag{A-3}$$

$p_0 = -2.99610\,70770\,35421\,74\,(-3)$	$q_0 = 1.06209\,23052\,84679\,18\,(-2)$
$p_1 = -4.94730\,91062\,32507\,34\,(-2)$	$q_1 = 1.91308\,92610\,78298\,41\,(-1)$
$p_2 = -2.26956\,59353\,96869\,30\,(-1)$	$q_2 = 1.05167\,51070\,67932\,07\,(0)$
$p_3 = -2.78661\,30860\,96477\,88\,(-1)$	$q_3 = 1.98733\,20181\,71352\,56\,(0)$
$p_4 = -2.23192\,45973\,41846\,86\,(-2)$	$q_4 = 1.00000\,00000\,00000\,00\,(0)$

The maximum relative error $\leqslant 10^{-15.61}$.

The reciprocal of the complete gamma function $\Gamma(a)$, when $a \leqslant 1$, is computed from (25), (see also (119)). The coefficients C_k which occur are given in [13]. The program uses at most the first seventeen with a maximum relative error in $1/\Gamma(a)$ of less than 5×10^{-14}. We have

$C_1 = 1.0$	$C_9 = -0.11651\,67591\,85907\,(-2)$
$C_2 = 0.57721\,56649\,01533$	$C_{10} = -0.21524\,16741\,14951\,(-3)$
$C_3 = -0.65587\,80715\,20254$	$C_{11} = 0.12805\,02823\,88116\,(-3)$
$C_4 = -0.42002\,63503\,40952\,(-1)$	$C_{12} = -0.20134\,85478\,07882\,(-4)$
$C_5 = 0.16653\,86113\,82291$	$C_{13} = -0.12504\,93482\,14267\,(-5)$
$C_6 = -0.42197\,73455\,55443\,(-1)$	$C_{14} = 0.11330\,27231\,98170\,(-5)$
$C_7 = -0.96219\,71527\,87697\,(-2)$	$C_{15} = -0.20563\,38416\,97761\,(-6)$
$C_8 = 0.72189\,43246\,66310\,(-2)$	$C_{16} = 0.61160\,95104\,48142\,(-8)$

$$C_{17} = 0.50020\,07644\,46922\,(-8)$$

It was stated on page 38 that H was obtained during the computation for $1/\Gamma(a)$. We have for

$$0 < a < 1/2, \qquad H = \sum_{2}^{17} C_k\, a^{k-1}$$

which is terminated when $k = 17$ or before if $k = n < 17$ with

$$\left| C_n \, a^{n-1} \right| < \frac{1}{2} \epsilon \left| \sum_2^n C_k \, a^{k-1} \right| .$$

Then

$$1/\Gamma(a) = a[1 + H] . \qquad 0 < a < 1/2 .$$

For $\quad \frac{1}{2} \leqslant a < 1, \quad \widetilde{H} = \sum_2^{17} C_k (a - 1)^{k-1}, \quad H = \frac{1}{a} (1 - a + \widetilde{H}) ,$

where $\widetilde{H}$ is terminated when $k = 17$ or before if $k = n < 17$ with

$$\left| C_n (a - 1)^{n-1} \right| < \frac{1}{2} \epsilon \left| \sum_2^n C_k (a - 1)^{k-1} \right| .$$

Then $\quad 1/\Gamma(a) = 1 + \widetilde{H} \qquad \frac{1}{2} \leqslant a < 1.$

COEFFICIENTS IN TAKENAGA'S METHOD

In this Appendix, we list the coefficients associated with the modified Takenaga procedure as used in 17 (see (17) or (115)). Also the coefficients for (18) or (80) are listed as well as those needed in (63). For (18) and (63) additional coefficients beyond those actually stored in the program are given. All of the coefficients were obtained initially in rational form by A. Morris, [5].

The factor C, which appears in (17) is found from $e^{\ln C}$, where the expansion of $\ln C$ in terms of $R = b$ is given by (63). The expansion of $\ln C$ to b^{-11} is

$$
\begin{aligned}
\ln C \cong\ & 1/36\,b \qquad\qquad (b = a - 1) \qquad\qquad \text{(B-1)}\\
& -1/81\,b^2\\
& +1/360\,b^3\\
& +2/1215\,b^4\\
& -691/306180\,b^5\\
& +16/15309\,b^6\\
& -373/3674160\,b^7\\
& +80/177147\,b^8\\
& -44069/38972340\,b^9\\
& +1792/9743085\,b^{10}\\
& +114953857/63836692920\,b^{11}.
\end{aligned}
$$

From the expression (B-1), we obtain C in terms of inverse powers of b (see (80)) as far as the eleventh power, i.e.,

$$C = e^{\ell n\, C} = 1 \qquad (\quad - 1) \qquad \textbf{(B-2)}$$

$+ 1/36\, b$

$- 31/2592\, b^2$

$+ 3413/1399680\, b^3$

$+ 361733/201553920\, b^4$

$- 113888281/50791587840\, b^5$

$+ 7565202533/7836416409600\, b^6$

$81332300683/1974776935219200\, b^7$

$+ 245886906474757/568735757343129600\, b^8$

$- 113361750483248635 33/1013487119585456947 2000\, b^9$

$+ 30902061509879955337/2043190033084281205555 20\, b^{10}$

$+ 4346156164844627985130999/2390532338708609010499584000\, b^{11}$

We proceed to give the coefficients $N_{2k,j}$, $M_{2k+1,j}$ for (17) or (115). Also see (68). We have from (115)

$$T(a, x) = Z(s)\, C \left| b_0 + \sum_{k=1}^{12} a^{-2k} \left(\sum_{j=k+1}^{2k} N_{2k,j}\, b_j \right) \right.$$

$$\left. + \sum_{k=1}^{12} a^{-(2k+1)} \left(\sum_{j=k+1}^{2k} M_{2k+1,j}\, a_j \right) \right| .$$

The a_j, b_j are defined by (109) and (89). Then listing the $N_{2k,j}$ first

$\underline{\underline{N_{2k,j}}}$

	2k = 0
j = 0	1.00000000000000

	2k = 2
j = 2	0.833333333333333 01

	2k = 4
j = 3	0.555555555555556D 01
4	0.347222222222222D 02

	2k = 6
j = 4	0.416666666666667D 01
5	0.685185185185185D 02
6	0.964506172839506D -04

	2k = 8
j = 5	0.333333333333333D 01
6	0.819003527336861D 02
7	0.378086419753086D 03
8	0.200938786008230D 05

$N_{2k,j}$ (continued)

2k = 10

j = 6	–0.277777777777778D 01
7	0.869551524313429D 02
8	–0.661283803644915D 03
9	0.130744170096022D 04
10	–0.334897976680384D 07

2k = 12

j = 7	0.238095238095238D 01
8	0.881859267275934D 02
9	–0.906422925823279D 03
10	0.315111821477562D 04
11	–0.325967363968907D 06
12	0.465136078722756D 09

2k = 14

j = 8	0.208333333333333D 01
9	0.875449995820366D 02
10	–0.110924409279237D 02
11	0.543384163221759D 04
12	–0.103601699095848D 05
13	0.632585067062948D 08
14	–0.553733427050900D 11

2k = 16

j = 9	–0.185185185185185D 01
10	0.859664255497589D 02
11	–0.127442064638934D 02
12	0.792846219268942D 04
13	–0.219833248895504D 05
14	0.256406041902548D 07
15	–0.100262665858016D 09
16	0.576805653178020D 13

2k = 18

j = 10	0.166666666666667D 01
11	0.839280174173222D 02
12	–0.140822229954083D 02
13	0.104859749386735D 03
14	0.377869445383984D 05
15	0.657099645757853D 07
16	–0.504835073886125D 09
17	0.134126541218996D 11
18	0.534079308498167D 15

2k = 20

j = 11	0.151515151515152D 01
12	0.816854779541828D 02
13	–0.151645284022835D 02
14	0.130144560538248D 03
15	0.571479941762549D 05
16	0.131587654927232D 06
17	0.153805654474373D 08
18	0.821350741480055D 11
19	–0.155096631187868D 13
20	0.445066090415139D 17

$\underline{\underline{N_{2k,j}}}$ (continued)

2k = 22

j = 12	-0.138888888888889D-01
13	0.793794836275189D-02
14	-0.160397649389170D-02
15	0.154597809511729D-03
16	-0.793808101128600D-05
17	0.226017561582738D-06
18	-0.3552506641860[illegible]41D-08
19	0.293512513022109D-10
20	-0.113515291507799D-12
21	0.157850106733903D-15
22	-0.337171280617530D-19

2k = 24

j = 13	-0.128205128205128D-01
14	0.770882874769910D-02
15	-0.167472688827663D-02
16	0.177919959495527D-03
17	-0.103838417905802D-04
18	0.349876079228262D-06
19	-0.688447027071384D-08
20	0.774945030060715D-10
21	-0.470327376028715D-12
22	0.136082899158372D-14
23	-0.143410184689323D-17
24	0.234146722651062D-21

$\underline{\underline{M_{2k+1,j}}}$

2k + 1 = 3

j = 2	0.666666666666667D-01

2k + 1 = 5

j = 3	0.476190476190476D-01
4	-0.555555555555556D-02

2k + 1 = 7

j = 4	0.370370370370370D-01
5	-0.767195767195767D-02
6	0.231481481481481D-03

2k + 1 = 9

j = 5	0.303030303030303D-01
6	-0.850970017636684D-02
7	0.523368606701940D-03
8	-0.643004115226337D-05

2k + 1 = 11

j = 6	0.256410256410256D-01
7	-0.878921490032601D-02
8	0.789241622574956D-03
9	-0.215681951793063D-04
10	0.133959190672154D-06

$M_{2k+1,j}$ (continued)

2k + 1 = 13

j	
7	0.222222222222222D−01
8	−0.880261713595047D−02
9	0.101291320868040D−02
10	−0.425382944673068D−04
11	0.624377408713828D−06
12	−0.223265317786923D−08

2k + 1 = 15

j	
8	0.196078431372549D−01
9	−0.868360035026702D −02
10	0.119614702703957D−02
11	−0.666557780571596D−04
12	0.156175357088423D−05
13	−0.137999229755917D−07
14	0.310090719148504D 10

2k + 1 = 17

j	
9	0.175438596491228D−01
10	−0.849851422913981D−02
11	0.134486033350672D −02
12	−0.920618640770127D −04
13	0.293982360735374D−05
14	−0.427076358950073D−07
15	0.245414654868959D−09
16	−0.369155618033933D 12

2k + 1 = 19

j	
10	0.158730158730159D−01
11	−0.828215615367318D−02
12	0.146520693061439D−02
13	−0.117579268958914D−03
14	0.470657699079954D−05
15	−0.952012948298324D−07
16	0.920927302909851D−09
17	−0.364022597059366D−11
18	0.384537102118680D−14

2k + 1 = 21

j	
11	0.144927536231884D−01
12	−0.805344609524176D−02
13	0.156253676162327D−02
14	−0.142498276043991D−03
15	0.679474426392697D−05
16	−0.175141331957715D−06
17	0.240079107041612D−08
18	−0.162881630964713D−10
19	0.462250219327808D−13
20	−0.356052872332111D−16

$M_{2k+1,j}$ (continued)

$2k + 1 = 23$		$2k + 1 = 25$	
j = 12	0.133333333333333D -01	j = 13	0.123456790123457D -01
13	-0.782287315947945D -02	14	-0.759626377172480D -02
14	0.164123530554189D -02	15	0.170482035596103D -02
15	-0.166412253953536D -03	16	-0.189106579970833D -03
16	0.913690962754997D -05	17	0.116720392056449D -04
17	-0.284271767098726D -06	18	-0.422734470127948D -06
18	0.503434278629405D -08	19	0.913761861504362D 08
19	-0.491182369134031D -10	20	-0.116534009325387D -09
20	0.243170716076481D -12	21	0.840151056535048D -12
21	-0.512512677374051D -15	22	-0.313185291085580D -14
22	0.296710726943426D -18	23	0.503843072514408D -17
		24	-0.224780853745020D -20

ASYMPTOTIC EXPANSION FOR $Q(x + 1, x)$ $(x \to \infty)$

In this appendix we show how the asymptotic expansion of (121) was obtained. The form of the expansion is given by

$$Q(x+1, x) \cong \frac{1}{2} + \frac{1}{3}\sqrt{\frac{2}{\pi x}} \sum_{k=0}^{M-1} e_k/x^k, \quad (x \to \infty), \qquad \text{(C-1)}$$

where

$$e_0 = 1, \quad e_1 = -23/180, \quad e_2 = 23/2016,$$

as shown on page 43. The e_k for $k = 0, 1, \cdots, M = 15$ are given on page C-8 of this appendix.

We start with

$$Q(x+1, x) = \Gamma(x+1, x)/\Gamma(x+1) \quad \text{(See (2), (4), (5))}, \qquad \text{(C-2)}$$

where the asymptotic expansions for the two functions on the right hand side are given by

$$1/\Gamma(x+1) = 1/x\,\Gamma(x) = e^{-\ln x\,\Gamma(x)} \cong e^x \left(\sqrt{2\pi}\, x^{x+1/2}\right)^{-1} \sum_{k=0}^{M-1} f_k/x^k, \qquad \text{(C-3)}$$

with

$$\ln x\,\Gamma(x) \cong \left(x + \frac{1}{2}\right) \ln x - x + \frac{1}{2}\ln 2\pi + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}, \quad (x \to \infty), \qquad \text{(C-4)}$$

(B_{2m}, the $2m^{th}$ Bernoulli number), (See [1], p. 257, 810),

and

$$\Gamma(x+1, x) \cong e^{-x} x^{x+1/2} \sum_{k=1}^{2M} b_k\, \Gamma(k/2)/x^{(k-1)/2}, \quad (x \to \infty). \qquad \text{(C-5)}$$

The coefficients $b_k \, \Gamma\left(\frac{k}{2}\right)$ and f_k are constants. The multiplication of the series in (C-3) and (C-5) to yield (C-1) was carried out by A. Morris using his "Flap" routine, [5]. In addition, he also used "Flap" before hand to determine the f_k and $b_k \, \Gamma\left(\frac{k}{2}\right)$ in rational form. For completeness they are included on pages C-6 and C-7, respectively.

The b_k were computed independently by M. Saizan in decimal form and are given below.

b_1 = .70710678118654752440084436210+00
b_2 = .66666666666666666666666666670+00
b_3 = .11785113019775792073347406040+00
b_4 = -.29629629629629629629629629620-01
b_5 = .32736425054932755759298350060-02
b_6 = .14109347442680776014109347450-02
b_7 = -.10111917961412562334538023690-02
b_8 = .31354105428179502253576327620-03
b_9 = -.24725527389373814204443595020-04
b_{10} = -.29664995371442559371228781380-04
b_{11} = .18773314722352835450708456900-04
b_{12} = -.56531048757843453773952173490-05
b_{13} = .30356279857570629915558755250-06
b_{14} = .65675582619137971472473326840-06
b_{15} = .39661662515866433810910363460-06
b_{16} = .11709055465263091499753584840-06
b_{17} = .46185624551259091628844910680-08
b_{18} = .14926776659329088172708068980-07
b_{19} = .88199673073439970634256291640-08
b_{20} = -.25741666714184579016143615190-08
b_{21} = .79680309541185471304773765760-10
b_{22} = .34526855806986090279195636110-09
b_{23} = .20163942557072502446885179830-09
b_{24} = .58439462516833163681724231240-10
b_{25} = -.14884457788493658882335740850-11
b_{26} = .80905372853655317853782999600-11
b_{27} = .46917438130337419702825947300-11
b_{28} = -.13535257572473374162248894010-11
b_{29} = .29316941779365506468896397500-13
b_{30} = .19147882067656010283127746780-12

We note the first three terms of (C-5) are given in [1, p. 263].

---

*The symbols b_k and a_k are used in this appendix to retain the notation in [6]. They should not be confused with those in (109).

The derivation below for (C-5) is given for completeness and is based on the material given in [6, p. 85–87].

We have

$$\Gamma(x + 1, x) = \int_x^\infty t^x e^{-t}\, dt. \tag{C-6}$$

Letting $t = x(1 + u)$, (C-6) transforms to

$$\Gamma(x + 1, x) = e^{-x} x^{x+1} \int_0^\infty e^{-x[u - \ln(1+u)]}\, du. \tag{C-7}$$

The integral on the right in (C-7) is treated in [6, p. 87]. Its asymptotic expansion $(x \to \infty)$ is obtained by a series reversion. Indeed, letting

$$V = u - \ln(1 + u) = \frac{u^2}{2} - \frac{u^3}{3} + \cdots \quad , \quad |u| < 1, \tag{C-8}$$

we note that

$$\frac{dV}{du} = 1 - \frac{1}{1 + u} = \frac{u}{1 + u}$$

or

$$\frac{du}{dV} = \frac{1 + u}{u} . \tag{C-9}$$

Then, assuming an expansion for u of the form

$$u = \sqrt{2V} + \sum_{k=2}^{\infty} a_k V^{k/2}, \quad (a_1 = \sqrt{2}), \tag{C-10}$$

we have by differentiating (C-10) and using (C-9)

$$\frac{1 + u}{u} = \sum_{k=1}^{\infty} \frac{k}{2} a_k V^{\frac{k}{2} - 1}, \quad a_1 = \sqrt{2} .$$

It follows from this result that

$$1 + \sum_{k=1}^{\infty} a_k V^{\frac{k}{2}} = \sum_{k=1}^{\infty} a_k V^{\frac{k}{2}} \sum_{k=1}^{\infty} \frac{k}{2} a_k V^{\frac{k}{2} - 1} \tag{C-11}$$

Expressing the product on the right as a Cauchy product, (C-11) becomes

$$1 + \sum_{k=1}^{\infty} a_k V^{\frac{k}{2}} = \sum_{i=1}^{\infty} V^{\frac{i}{2} - 1} \sum_{j=1}^{i} \frac{j}{2} a_j a_{i-j} \tag{C-12}$$

which contains a linear set of equations for the a_k, $k = 0, 1, \cdots$ with $a_0 \equiv 0$. For example, equating coefficients of like powers of V, we have

$$V^{-1/2}: \qquad 0 = \frac{1}{2} a_1 a_0 \Rightarrow a_0 = 0, \text{ since } a_1 = \sqrt{2}$$

$$V^{0}: \qquad 1 = \frac{1}{2} a_1^2 + a_2 a_0 \Rightarrow a_1 = \sqrt{2}$$

$$V^{1/2}: \qquad a_1 = \frac{1}{2} a_1 a_2 + a_1 a_2 \Rightarrow a_2 = 2/3$$

$$\vdots$$

In general, by setting $k = n - 1$, $i = n + 1$, (C-12) can be solved for a_n. We get

$$a_{n-1} = \sum_{j=1}^{n} \frac{j}{2} a_j \cdot a_{n+1-j} \qquad (a_0 = 0)$$

or

$$a_n = \frac{2}{(n+1)a_1} \left[a_{n-1} - \sum_{j=2}^{n-1} \frac{j}{2} a_j a_{n+1-j} \right], \qquad n = 3, 4, \cdots . \tag{C-13}$$

Thus,

$$a_2 = 2/3, \quad a_3 = \sqrt{2}/18, \quad a_4 = \;2/135, \quad a_5 = \sqrt{2}/1080, \quad a_6 = 4/8505,$$

etc.

The expression (C-13) gives the coefficients for the inverted series in (C-10). Therefore, following [6], the coefficients b_k of asymptotic series (C-5) can now be determined. We have

$$\frac{du}{dV} = \frac{d}{dV}\left(\sum_{k=1}^{\infty} a_k\, V^{k/2}\right) = \sum_{k=1}^{\infty} \left(\frac{k}{2}\right) a_k\, V^{(k-2)/2}$$

$$= \sum_{k=1}^{\infty} b_k\, V^{(k-2)/2}, \quad b_k = \frac{k}{2}\, a_k\, .$$

The b_k are the desired coefficients that appear in (C-5). Thus, e.g.,

$$b_1 = \frac{1}{2}\, a_1 = 1/\sqrt{2}, \quad b_2 = a_2 = 2/3, \quad b_3 = \frac{3}{2}\, a_3 = \frac{\sqrt{2}}{12}, \quad b_4 = 2a_4 = -\,4/135.$$

The a_k are listed below for completeness.

a_1	=	.1414213562373095048801688724D+01
a_2	=	.6666666666666666666666666667D+00
a_3	=	.7856742013183861382231604023D-01
a_4	=	.1481481481481481481481481481D-01
a_5	=	.1309457002197310230371934003D-02
a_6	=	.4703115814226925338036449149D-03
a_7	=	.2889119417546446381296806769D-03
a_8	=	.7838526357044875563394031905D-04
a_9	=	.5494561642083069823209687782D-05
a_{10}	=	.5932999074288511874245756276D-05
a_{11}	=	.3413329949518697354674264830D-05
a_{12}	=	.9421841459640575628992028915D-06
a_{13}	=	.4670196901164712294701346961D-07
a_{14}	=	.9382226088448281638924760977D-07
a_{15}	=	.5286221666782191174788048462D-07
a_{16}	=	.1463631933157886437469198105D-07
a_{17}	=	.5433602888383422544570483138D-09
a_{18}	=	.1656530739925454241412007665D-08
a_{19}	=	.9284176113625260066763820173D-09
a_{20}	=	.2574166671418457901614361519D-09
a_{21}	=	.7588600908684330600454644358D-11
a_{22}	=	.3138805073362371843563239646D-10
a_{23}	=	.1753386309310652386685667811D-10
a_{24}	=	.4869955209736096973477019270D-11
a_{25}	=	.1190756623079492710586859268D-12
a_{26}	=	.6223490219511947527214076892D-12
a_{27}	=	.3475365787432401459468588689D-12
a_{28}	=	.9668041123195267258749210007D-13
a_{29}	=	.2021658053750724721992388948D-14
a_{30}	=	.1276525471177067352208516452D-13

The rational coefficients f_k for (C-3) are included in the expression below, (C-14). The quantity $T \equiv 1/\sqrt{x}$. The coefficient of $1/x^3$, f_3, is given by 139/51840.

```
1

-1/12*T**2

1/288*T**4

139/51840*T**6

-571/2488320*T**8

-163879/209018880*T**10

5246819/75246796800*T**12

534703531/902961561600*T**14

-4483131259/86684309913600*T**16

-432261921612371/514904800886784000*T**18

6232523202521089/86504006548979712000*T**20

25834629665134204969/13494625021640835072000*T**22

-1579029138854919086429/9716130015581401251840000*T**24

-746590869962651602203151/116593560186976815022080000*T**26

15[illegible]1513601028097903631961/2798245444487443560529920000*T**28
```

(C-14)

The coefficients $b_k\,\Gamma(k/2)$ for (C-5) are included in the expression below, (C-15). The quantity $T \equiv 1/\sqrt{x}$. The coefficient of $1/x^{3/2}$, $b_4\,\Gamma(2)$, is given by $-4/135$. The coefficient of $1/x^3$, $b_7\,\Gamma(7/2)$, is given by $(-139/103680)\sqrt{2\pi}$.

```
1/2*2**(1/2)*PI**(1/2)
2/3*T
1/24*2**(1/2)*PI**(1/2)*T**2
-4/135*T**3
1/576*2**(1/2)*PI**(1/2)*T**4
8/2835*T**5
-139/103680*2**(1/2)*PI**(1/2)*T**6
16/8505*T**7
-571/4976640*2**(1/2)*PI**(1/2)*T**8
-8992/12629925*T**9
163879/418037760*2**(1/2)*PI**(1/2)*T**10
-334144/492567075*T**11
5246819/150493593600*2**(1/2)*PI**(1/2)*T**12
698752/1477701225*T**13
-534703531/1805923123200*2**(1/2)*PI**(1/2)*T**14
23349012224/39560545029375*T**15
-4483131259/173368619827200*2**(1/2)*PI**(1/2)*T**16
-1357305243136/2255230667064375*T**17
432261921612371/1029809601773568000*2**(1/2)*PI**(1/2)*T**18
-6319924923392/6765692001193125*T**19
6232523202521089/173008013097959424000*2**(1/2)*PI**(1/2)*T**20
8773495082018816/70032491221234834375*T**21
-25834629665134204969/26989250043281670144000*2**(1/2)*PI**(1/2)*T**22
43904477022654464/21007473663704653125*T**23
-1579029138854919086429/19432260031162802503680000*2**(1/2)*PI**(1/2)*T**24
-170965059433760358784/4411569460937797715625*T**25
746590869962651602203151/233187120373953630044160000*2**(1/2)*PI**(1/2)*T**26
-480330505318434367035260928/56995271759028775870171875*T**27
1511513601028097903631961/5596490888974887121059840000*2**(1/2)*PI**(1/2)*T**28
906481173388026066736939006/530056027364547615592584375*T**29
```

(C-15)

The coefficients e_k for (C-1) are included in rational form in (C-16) below and then e_k, $k \geqslant 0$, are given in decimal form in (C-17). Again $T = 1/\sqrt{x}$. The coefficient of $1/x^{3/2}$ from (C-16) is $(\ 23/270)/\sqrt{2\pi}$.

```
1/2
2/3*2**(-1/2)*PI**(-1/2)*T
-23/270*2**(-1/2)*PI**(-1/2)*T**3
23/3024*2**(-1/2)*PI**(-1/2)*T**5
259/77760*2**(-1/2)*PI**(-1/2)*T**7
-2016373/1847577600*2**(-1/2)*PI**(-1/2)*T**9
-4568387/4075868160*2**(-1/2)*PI**(-1/2)*T**11
67839059/112870195200*2**(-1/2)*PI**(-1/2)*T**13
11321196179287/12088397905920000*2**(-1/2)*PI**(-1/2)*T**15
-1740670460353/2470502832537600*2**(-1/2)*PI**(-1/2)*T**17
-1109305762374011/772357201330176000*2**(-1/2)*PI**(-1/2)*T**19
628733540424335252233/44765823389097600960000*2**(-1/2)*PI**(-1/2)*T**21
14145736624903202333/40483875064922505216000*2**(-1/2)*PI**(-1/2)*T**23
-432054907011862914451563/10201936516360471314432000000*2**(-1/2)*PI**(-1/2)*T**25
-93022993595952700027436859007/753106525041762346063688128000000*2**(-1/2)*PI**(-1/2)*T**27
2334609624077295573646030647/13011841316866612556466412800000*2**(-1/2)*PI**(-1/2)*T**29
```

(C-16)

Decimal form for e_k, $k \geqslant 0$.

e_0	=	1.00000	00000	00000	00000	00000	00000F	00
e_1	=	−1.27777	77777	77777	77777	77777	77777F	−01
e_2	=	1.14087	30158	73015	87301	58730	15873F	−02
e_3	=	4.99614	19753	08641	97530	86419	75308F	03
e_4	=	1.63704	05768	07166	31333	91528	45325F	03
e_5	=	1.68125	66626	29637	16667	42037	11240F	03
e_6	=	9.01554	11107	14762	01363	03263	87528F	04
e_7	=	1.40480	10663	69107	33117	86737	76069F	03
e_8	=	1.05651	27415	16873	01128	36110	56040F	03
e_9	=	2.15439	00162	97250	29517	66213	75916F	03
e_{10}	=	2.10674	16054	60398	61407	67106	91024F	03
e_{11}	=	5.24124	92389	44466	72838	31327	57108F	03
e_{12}	=	6.35254	25734	46898	69625	71691	54688F	03
e_{13}	=	1.85264	11158	31105	51190	42883	84580F	02
e_{14}	=	2.69132	88833	12297	39720	54734	83920F	02

(C-17)

CRITERION FOR TERMINATING THE SERIES IN (10)

In the main text of this report an expression for P(a, x) is given by (10), i.e.,

$$P(a, x) = \frac{e^{-x} x^a}{a\Gamma(a)} \left[1 + \frac{x}{a+1} + \frac{x^2}{(a+1)(a+2)} + \cdots + s_n + T_{N+1} \right], \qquad \text{(D-1)}$$

where

$$s_n \equiv x^n / [(a+1)(a+2) \cdots (a+n)] \qquad \text{(D-2)}$$

$$T_{n+1} \equiv s_{n+1} + s_{n+2} + \cdots . \qquad \text{(D-3)}$$

The objective in this Appendix is to prove the statement made in Section 5 dealing with (90). Given an $\epsilon > 0$, if N is defined as the smallest positive integer for which $s_N \leqslant \frac{\epsilon}{2}$, we wish to show the truncation error, T_{N+1}, associated with the series in square brackets of (D-1) is less than ϵ, provided $x/(a+N+1) \leqslant 2/3$. The proof requires only a few lines.

From (D-3)

$$T_{N+1} = s_N \left(\frac{x}{a+N+1} \right) \left[1 + \frac{x}{a+N+2} + \frac{x^2}{(a+N+2)(a+N+3)} + \cdots \right]$$

$$< \frac{\epsilon}{2} \cdot \frac{2}{3} \cdot \sum_{k=0}^{\infty} (2/3)^k = \epsilon$$

where we have used the inequalities $x/(a+N+j) < x/(a+N+1) \leqslant 2/3, j = 2, 3, \cdots$.

FORTRAN LISTING OF THE PROGRAM

Calling sequence to the Incomplete Gamma Function Ratio Subroutine.

CALL PAX(a, x, P, Q, IOP) where

$$P = \int_0^x e^{-t}\, t^{a-1}\, dt \Big/ \int_0^\infty e^{-t}\, t^{a-1}\, dt$$

$$Q = 1 - P$$

IOP = 0 for 12 significant digits of accuracy with an error no greater than 1 unit in the 12th significant digit of P and Q.

IOP = 1 for 6 significant digits of accuracy with an error no greater than 1 unit in the 6th significant digit of P and Q.

IOP = 3 for 3 significant digits of accuracy with an error no greater than 1 unit in the 3rd significant digit of P and Q.

The routine is designed in most cases to set P = 0, Q = 1 if $P \leqslant \bar{\epsilon}$ and P = 1, Q = 0 if $Q \leqslant \bar{\epsilon}$. The quantity $\bar{\epsilon}$ is ordinarily set to take the specified value $\epsilon = 5 \times 10^{-13}$, 5×10^{-7}, or 5×10^{-4} when IOP is set to 0, 1, 3, respectively. However, $\bar{\epsilon}$ may be set internally to zero or any other positive value less than the value of ϵ specified by IOP. In the program $\bar{\epsilon}$ is stored in ACO.

Restrictions: $x \geqslant 0$ and $a > 0$.

Error Returns: If either $a \leqslant 0$ and/or $x < 0$ then P is set equal to 2.

```
      SUBROUTINE PAX  ( A,X,ANS,QANS,IOP3)
      DIMENSION VA(20),VB(20),ZA(30),ZB(30) ,SA(20),SB(20)
      DATA
     1 P0/2.4266 79552 30532 E+2   /,
     2 P1/2.1979 26161 82942 E+1   /,
     3 P2/6.9963 63488 61914 E+0   /,
     4 P3/-3.5609 84370 18154 E-2  /,
     5 Q0/2.1505 88758 69861 E+2   /,
     6 Q1/9.1164 90540 45149 E+1   /,
     7 Q2/1.5082 79763 04078 E+1   /,
     8 Q3/1.                       /,
     9 C0/3.0045 92610 20162 E+2   /
      DATA
     1 C1/4.5191 89537 11873 E+2   /,
     2 C2/3.3932 08167 34344E+2    /,
     3 C3/1.5298 92850 46940 E+2   /,
     4 C4/4.3162 22722 23567 E+1   /,
     5 C5/7.2117 58250 88309 E+0   /,
     6 C6/5.6419 55174 78974 E-1   /,
     7 C7/-1.3686 4857 3 82717E-7  /,
     8 ALPH0/-2.9961 07077 03542 E-3/,
     9 ALPH1/-4.9473 09106 23251 E-2  /
      DATA
     1 ALPH2/-2.2695 65935 39687 E-1 /,
     2 ALPH3/-2.7866 13086 09648 E-1 /,
     3 ALPH4/-2.2319 24597 34185 E-2 /,
     4 BET0/1.0620 92305 28468 E-2    /,
     5 BET1/1.9130 89261 07830 E-1    /,
     6 BET2/1.0516 75167 06793        /,
     7 BET3/1.9873 32018 17135        /,
     8 BET4/1.          /,
     9 D0/3.0045 92609 56983 E+2      /
      DATA
     1 D1/7.9095 09253 27898 E+2   /,
     2 D2/9.3135 40948 50610 E+2   /,
     3 D3/6.3898 02644 65631 E+2   /,
     4 D4/2.7758 54447 43988 E+2   /,
     5 D5/7.7000 15293 52295 E+1   /,
     6 D6/1.2782 72731 96294 E+1   /,
     7 D7/1.         /
C        CST2=SP(DP3413./DP13996 80. )
C        CST3=SP(DP361733./DP20155 3920. )
C        CST4=SP(DP(11383 8281./DP50791 58784 0. )
C        CST5=SP(DP75652 02533./DP78364 16409 600. )
      DATA
     1 CST5/.96539 05736 46935E-3  /,
     2 CST4/.22422 66600 50012E-2  /,
     3 CST3/.17947 20737 75533E-2  /,
     4 CST2/.24334 14494 74166E-2  /,
     5 GAMPT5/1.7724 53850 90552   /,
     6 RT2PI/2.5066 28274 63100    /
     7,RTPI/1.7724 53850 90552     /
      DATA ACC1/5.E-13/,ACC3/5.E-7/,ALPHA3/4.892/,
     1   ALPHA1/7.1306/,    AC0/5.E-13/,
     1   X01/31.000/,X03/17.00/,CEP/2.3025 85092 99405/
```

```
2      ,UX1/1.0366/,UX3/2.9245/
3      ,RT2/.14142 13562 37310E+1/,ABAR/100./
4        ,ALPHA5/3.29053/,X05/9.7/,UX5/3.887/,ACC5/5.E-4/
      ASAV=A
      INO=0
      ANS=0.
      IF ( IOP3.NE.0 )  GO TO 1105
      ALPHA=ALPHA1
      X0=X01
      ACC=ACC1
      UX=UX1
      GO TO 1111
1105  CONTINUE
      IF ( IOP3.NE.1 )  GO TO 1107
      ALPHA=ALPHA3
      ACC=ACC3
      X0=X03
      UX=UX3
      GO TO 1111
1107  CONTINUE
      ALPHA=ALPHA5
      ACC=ACC5
      X0=X05
      UX=UX5
1111  CONTINUE
       IF ( A.LE.0.0 .OR.X.LT.0.0 )  GO TO 1131
      IF ( X.GT.0.0 )  GO TO 1151
1121  CONTINUE
      GO TO 1171
1131  CONTINUE
      ANS=2.
      RETURN
1151  CONTINUE
      IF ( A.LT.15. )  GO TO 1331
      RTA=SQRT(A)
      IF ( ACO.LT.ACC1 )  GO TO 1155
      IF ( A.GE.X )  GO TO 1161
      FP=1.-1./(9.*A)+ALPHA/(3.*RTA)
      FP=-FP*FP*FP*A+X
      IF ( FP.GE.0.0 )  GO TO 1191
1155  CONTINUE
      IF ( A.LT.ABAR )  GO TO 1331
      IF ( (.75*A).GT.X.OR.(1.25*A).LT.X )  GO TO 1331
      GO TO 5011
1161  CONTINUE
      IF ( X.LE.UX )  GO TO 1171
      FM=1.-1./(9.*A)-ALPHA/(3.*RTA)
      FM=-FM*FM*FM*A+X
      IF ( FM.LE.0.0  )  GO TO 1121
      GO TO 1155
1171  CONTINUE
      ANS=0.
      QANS=1.
      RETURN
1191  CONTINUE
```

```
 1311   CONTINUE
        ANS=1.
        QANS=0.
        RETURN
 1331   CONTINUE
        IF ( A.LE.X )  GO TO 1441
 1341   CONTINUE
        IF ( A.GE.30. )  GO TO 1343
        ALGX=ALOG(X)
        T1=-X+A*ALGX
        CALL GAMCO3 ( A,H,ACC,GANS )
        R=EXP(T1)/GANS
        GO TO 1349
 1343   CONTINUE
        AISQ=1./(A*A)
        T1=A-X+A*ALOG(X/A)+(((AISQ-1.3333 33333 33333)*AISQ+4.666 66666
      1  666667)*AISQ-140.)/(1680.*A)
        R=RTA*EXP(T1)*.39894 22804 01433
 1349   CONTINUE
        IF ( A.GT.X)  GO TO 1351
 1350   CONTINUE
        IF ( A.LT.1. )  GO TO 1459
        IF ( A.GE.X )  GO TO 3011
        GO TO 3005
 1351   CONTINUE
        IF ( R.LE.(A*(1.-X/(A+1.))*ACO))  GO TO 1171
        GO TO 1350
 1441   CONTINUE
        IF ( X.GE.X0 )  GO TO 1341
        TWOA=2.*A
        J=INT(TWOA)
        T1=ABS(FLOAT(J)-TWOA)
        IF ( T1.GT.0.)GO TO 1341
        I=J/2
        AF=A-FLOAT(I)
        IF ( AF.GT.0. )  GO TO 6011
        GO TO 6071
 1459   CONTINUE
        IF ( X.LT.1.5)  GO TO 1461
        T1=(2.+X-A)/(1.+X)
        IF ( R.LE.(ACO*X*T1) )  GO TO 1311
        IF ( R.GT.(.101*X*T1) )  GO TO 2011
        GO TO 7011
 1461   CONTINUE
        INO=1
        GO TO 3011
C
C         THIS IS PART F
C
 2011   CONTINUE
        ACC7=.5*ACC
        T7=A*ALGX
        CEE=2.
        T=T7/CEE
        SUM=T
```

```
 2031   CONTINUE
        CEE=CEE+1.
        T=(T*T7)/CEE
        SUM=SUM+T
        IF ( ABS(T).GT.ACC7 )  GO TO 2031
        T2=T7*(1.+SUM)
        TK=-X
        CEE=1.
        AJK=TK/(A+CEE)
        SUM=AJK
 2071   CONTINUE
        CEE=CEE+1.
        TK=-(X*TK)/CEE
        AJK=TK/(A+CEE)
        SUM=SUM+AJK
        IF ( ABS(AJK).GE.-ACC*SUM)  GO TO 2071
        T3=-A*SUM
        T1=H
        T4=T1+T2
        T5=T1*T2
        QANS=T3-T4+T3*(T4+T5)-T5
        ANS=1.-QANS
        RETURN
 3005   CONTINUE
        IF ( X.LE.CEP )  GO TO 3011
        IF ( X.LT.X0 )  GO TO 7011
        IF ( A.LE.2. )  GO TO 3006
        IF ( (R*(X+1.)).LE.(ACO*X*(2.+X-A)) )  GO TO 1311
        GO TO 9011
 3006   CONTINUE
        IF ( R.LE.(ACO*(X+1. -A))) GO TO 1311
        GO TO 9011
C
C
C         THIS IS PART A
C
 3011   CONTINUE
        HAVACC=ACC*.5
        ROVA=R/A
        SNP1=1.
        SUM=SNP1
        CEE=A
 3031   CONTINUE
        CEE=CEE+1.
        T1=X/CEE
        SNP1=SNP1*T1
        SUM=SUM+SNP1
        IF ( T1.GT..66666 66666 66667 )  GO TO 3031
 3041   CONTINUE
        IF ( SNP1.LE.HAVACC )  GO TO 3051
        CEE=CEE+1.
        SNP1=(SNP1*X)/CEE
        SUM=SUM+SNP1
        GO TO 3041
 3051   CONTINUE
```

```
      ANS=ROVA*SUM
      QANS=1.-ANS
      IF ( INO.EQ.1.AND.ANS.GT.0.9 )  GO TO 2011
      RETURN
C
C      THIS IS PART E1
C
 5011 CONTINUE
      ASAV=A
      A=A-1.
      ALPHY=3.*SQRT(A+2./3.)
      ALPINV=1./ALPHY
      A2THRD=A+.66666 66666 66667
      DELTA=(X-A)-.66666 66666 66667
      SZ=DELTA/A2THRD
      Z=EXP(.33333 33333 33333*ALOG(X/A2THRD))
      S0=(ALPHY*SZ)/( (Z+1.)*Z+1.)
      XSAV=X
      X=(S0*S0)/2.
      RTX=ABS(S0)/RT2
      T5=EXP(-X)
      IF ( X.GT.0.25 )  GO TO 5361
      T1=((P3*X+P2)*X+P1)*X+P0
      T3=((Q3*X+Q2)*X+Q1)*X+Q0
      DEL=RTX*(T1/T3)
      CERF=1.-DEL
      GO TO 5381
 5361 CONTINUE
      IF ( X.GT.16.)  GO TO 5371
      T1=C0+RTX*(C1+RTX*(C2+RTX*(C3+RTX*(C4+RTX*(C5+RTX*(C6+C7*RTX)))))
     1 )
      T3=D0+RTX*(D1+RTX*(D2+RTX*(D3+RTX*(D4+RTX*(D5+RTX*(D6+RTX*D7)))))
     1 )
      CERF=T1/T3
      DEL=1.-CERF
      GO TO 5381
 5371 CONTINUE
      T=1./X
      T1=ALPH0+T*(ALPH1+T*(ALPH2+T*(ALPH3+T*ALPH4)))
      T3=BET0+T*(BET1+T*(BET2+T*(BET3+T*BET4)))
      CERF=(1./RTX)*(1./RT PI+T1/(T3*X))
      DEL=1.-CERF
 5381 CONTINUE
      ZA0=+1.
      IF ( X.LE.0.25 )  ZB0=(.5*CERF*RT2PI)/T5
      IF ( X.GT.0.25 )  ZB0=.5*CERF*RT2PI
      IF ( S0.GE.0. )  GO TO 5389
      ZA0=-1.
 5389 CONTINUE
      X=XSAV
      ZB(1)=ZB0
      ZA(1)=ZA0
      S0SQ=S0*S
      T7=-1./(ABS(S0))
      TOL=ACC*ZB0
```

```
      DO 5391  L=2,5
      K=2*(L-1)
      T7=T7*S0SQ
      ZB(L)=-T7
     1   +(FLOAT(K)-1.)*ZB(L-1)
      ZA(L)=-T7    *S0+FLOAT(K)*ZA(L-1)
 5391 CONTINUE
      VB(1)=ZB(1)
      VB(2)= -.83333 33333 33333E-1*ZB(3)
      VA(1)=.66666 66666 66667E-1*ZA(3)
      SB(1)=VB(1)
      ALPHSQ=ALPINV*ALPINV
      ALPHCB=ALPHSQ*ALPINV
      SB(2)=VB(2)*ALPHSQ
      SA(1)=VA(1)*ALPHCB
      T15=SB(1)+SB(2)
      T17=SA(1)
      VB(3)= -(ZB(4)-.0625*ZB(5))*.55555 55555 55556E-1
      VA(2)=.47619 04761 90476E-1*ZA(4)-.55555 55555 55556E-2*ZA(5)
      T23=ALPHSQ
      I5=3
      MF=0
 5395 CONTINUE
      I7=I5-1
      T23=T23*ALPHSQ
      SB(I5)=VB(I5)*T23
      SA(I7)=VA(I7)*T23*ALPINV
      T15=T15+SB(I5)
      T17=T17+SA(I7)
      IF ( ABS(SB(I5)).LT.TOL.AND.ABS(SA(I7)).LT.TOL )  GO TO 5431
      I3=2*I5
      I5=I5+1
      T7=T7*S0SQ
      T25=4.*FLOAT(I5)-7.
      ZB(I3)=-T7+T25*ZB(I3-1)
      ZA(I3)=-T7*S0+(T25+1.)*ZA(I3-1)
      T7=T7*S0SQ
      ZB(I3+1)=-T7+(T25+2.)*ZB(I3)
      ZA(I3+1)=-T7*S0+(T25+3.)*ZA(I3)
      MF=MF+1
      GO TO ( 5397,5399,5401,5403,5405,5407,5409,5411,5413,5415,5431),MF
     1  F
 5397 CONTINUE
      VB(4)=-(ZB(5)-.16444 44444 44444*ZB(6)+.23148 14814 81481E-2*ZB(7
     1   )) *.41666 66666 66667E-1
      VA(3)=(ZA(5)-.20714 28571 42857*ZA(6)+.00625*ZA(7))*.37037 03703
     1    70370E-1
      GO TO 5395
 5399 CONTINUE
      VB(5)=-(ZB(6)-.24570 10582 01058*ZB(7)+.11343 59259 25926E-1*ZB(8)
     1    )-.60281 63580 24691E-4*ZB(9))*.33333 33333 33333E-1
      VA(4)=.30303 03030 30303E-1*ZA(6)-.45097 00176 36694E-2*ZA(7)
     1   +.52336 86067 01940E-3*ZA(8)-.64300 41152 26337E-5*ZA(9)
      GO TO 5395
 5401 CONTINUE
```

```
      VB(6)=-(ZB(7)-.31303 85487 52634*ZB(8)+.23876 21693 12169E-1*ZB(9
     1       )-.47067 90123 45679E-3*ZB(10)+.12056 32716 04938E-5*ZB(11))*
     2       .27777 77777 77778E-1
      VA(5)=.25641 02564 10256E-1*ZA(7)-.87892 14900 32601E-2*ZA(8)+
     1       .78924 16225 74956E-3*ZA(9)-.21568 19517 93063E-4*ZA(10)
     2      +ZA(11)*.13395 91906 72154E-6
      GO TO 5395
 5403 CONTINUE
      VA(6)=+.22222 22222 22222E-1*ZA(8)-.88026 17135 95047E-2*ZA(9)+
     1     .10129 13208 68040E-2*ZA(10)-.42538 29446 73060E-4*ZA(11)+
     2     .62437 74037 13828E-6*ZA(12)-.22326 53177 86923E-8*ZA(13)
      VB(7)=-.23809 52380 95238E-1*ZB(8)+.88185 92672 75934E-2*ZB(9)-
     1     .90642 29259 23279E-3*ZB(10)+.31511 18214 77562E-4*ZB(11)-
     2     .32596 73639 68907E-6*ZB(12)+.46513 60787 22756E-9*ZB(13)
      GO TO 5395
 5405 CONTINUE
      VA(7)=+.19607 84313 72549E-1*ZA(9)-.86836 00350 26702E-2*ZA(10)+
     1     .11961 47327 33957E-2*ZA(11)-.66655 77805 71596E-4*ZA(12)+
     2     .15617 53573 88423E-5*ZA(13)-.13799 92297 55917E-7*ZA(14)+
     3     .31009 07191 48504E-10*ZA(15)
      VB(8)=-.20833 33333 33333E-1*ZB(9)+.87544 99958 20366E-2*ZB(10)-
     2     .11092 44092 79237E-2*ZB(11)+.54338 41632 21759E-4*ZB(12)-
     3     .10360 16990 95848E-5*ZB(13)+.63258 50670 62948E-8*ZB(14)-
     3     .55373 34270 50900E-11*ZB(15)
      GO TO 5395
 5407 CONTINUE
      VA(8)=+.17543 85964 91228E-1*ZA(10)-.84985 14229 13981E-2*ZA(11)+
     1     .13448 60333 50672E-2*ZA(12)-.92061 86407 70127E-4*ZA(13)+
     2     .29398 23607 35374E-5*ZA(14)-.42707 63589 50073E-7*ZA(15)+
     3     .24541 46548 68959E-9*ZA(16)-.36915 56180 33933E-12*ZA(17)
      VB(9)=-.18518 51851 85185E-1*ZB(10)+.35966 42554 97589E-2*ZB(11)-
     1     .12744 20646 38934E-2*ZB(12)+.79284 62192 68942E-4*ZB(13)-
     2     .21983 32488 95504E-5*ZB(14)+.25640 60419 02548E-7*ZB(15)-
     3     .10026 26658 58016 E-9*ZB(16)+.57680 56531 78021E-13*ZB(17)
      GO TO 5395
 5409 CONTINUE
      VA(9)=+.15873 01587 30159E-1*ZA(11) -.82821 56153 67318E-2*ZA(12)
     1    +.14652 06930 61439E-2*ZA(13)-.11757 92689 58914E-3*ZA(14)+
     2     .47065 76990 79954E-5*ZA(15)-.95201 29482 98324E-7*ZA(16)+
     3     .92092 73029 39851E-9*ZA(17)-.36402 25970 59366E-11*ZA(18)+
     4     .38453 71021 18680E-14*ZA(19)
      VB(10)=-.16666 66666 66667E-1*ZB(11)+.83928 01741 73222E-2*ZB(12)
     1    -.14082 22299 54063E-2*ZB(13)+.10485 97493 86735E-3*ZB(14)-
     2     .37786 94453 83984E-5*ZB(15)+.65709 96457 57853E-7*ZB(16)-
     3     .50483 50738 86125E-9*ZB(17)+.13412 65412 18996E-11*ZB(18)-
     4     .53407 93084 98167E-15*ZB(19)
      GO TO 5395
 5411 CONTINUE
      VA(10)=+.14492 75362 31884E-1*ZA(12)-.80534 46095 24176E-2*ZA(13)
     2    +.15625 36761 62327E-2*ZA(14)-.14249 82760 43991E-3*ZA(15)+
     3     .67947 44263 92697E-5*ZA(16)-.17514 13319 57715E-6*ZA(17)+
     4     .24007 91070 41612E-8*ZA(18)-.16288 16309 64713E-10*ZA(19)+
     5     .46225 02193 27808E-13*ZA(20)-.35605 28723 32111E-16*ZA(21)
      VB(11)=-.15151 51515 15152E-1*ZB(12)+.81685 47795 41828E-2*ZB(13)
     1    -.15164 52840 22835E-2*ZB(14)+.13014 45605 38248E-3*ZB(15)-
```

```
     2   .57147 59417 62549E-5*ZB(16)+.13158 76549 27232E-6*ZB(17)-
     3   .15380 56544 74373E-8*ZB(18)+.82135 07414 80055E-11*ZB(19)-
     4   .15509 66311 87868E-13*ZB(20)+.44506 60954 15139E-17*ZB(21)
      GO TO 5395
 5413 CONTINUE
      VA(11)=+.13333 33333 33333E-1*ZA(13)-.78228 73159 47945E-2*ZA(14)
     1 +.16412 35305 54189E-2*ZA(15)-.16641 22539 53536E-3*ZA(16)+
     2   .91369 09627 54997E-5*ZA(17)-.28427 17670 98726E-3*ZA(18)+
     3   .50343 42785 29405E-8*ZA(19)-.49118 23691 34031E-10*ZA(20)+
     4   .24317 07160 76481E-12*ZA(21)-.51251 26773 74051E-15*ZA(22)+
     5   .29671 57269 43426E-18*ZA(23)
      VB(12)=-.13888 88888 88889E-1*ZB(13)+.79379 48362 75139E-2*ZB(14)
     1 -.16039 76493 89170E-2*ZB(15)+.15459 78095 11729E-3*ZB(16)-
     2   .79380 81011 28600E-5*ZB(17)+.22501 75615 82738E-6*ZB(18)-
     3   .35525 06641 86041E-8*ZB(19)+.29351 25130 22179E-10*ZB(20)-
     4   .11351 52915 07799E-12*ZB(21)+.15785 01067 33903E-15*ZB(22)-
     6   .33717 12806 17530E-19*ZB(23)
      GO TO 5395
 5415 CONTINUE
      VA(12)=+.12345 67901 23457E-1*ZA(14)  -.75962 63771 72480E-2*ZA(1
     1   5)+.17048 20355 96103E-2*ZA(16)-.18910 65799 70833E-3*ZA(17)+
     2   .11672 03920 56449E-4*ZA(18)-.42273 44701 27948E-6*ZA(19)+
     2   .91376 18615 34362E-8*ZA(20)-.11653 40093 25387E-9*ZA(21)+
     3   .84015 10565 35948E-12*ZA(22)-.31318 52910 8558.E-14*ZA(23)+
     4   .50384 30725 14418E-17*ZA(24)-.22478 98537 45920E-2 *ZA(25)
      VB(13)=-.12820 51282 05128E-1*ZB(14)+.77088 28747 69910E-2*ZB(15)-
     1 - .16747 26835 27663E-2*ZB(16)+.17791 99594 95527E-3*ZB(17)-
     2   .10383 64179 05802E-4*ZB(18)+.34987 60792 28262E-6*ZB(19)-
     3   .68844 70270 71384E-8*ZB(20)+.77434 59300 60715E-10*ZB(21)-
     4   .47032 73760 28715E-12*ZB(22)+.13608 28991 58372E-14*ZB(23)-
     5   .14341 71846 89323E-17*ZB(24)+.23414 67226 51062E-21*ZB(25)
      GO TO 5395
 5431 CONTINUE
      T=1./A
      CST=(((((CST5*T-CST4)*T+CST3)*T+CST2)*T-.11959 87654 32099E-1)*T
     1    +.27777 77777 77778 E-1)*T+1.
      PAP1X=(CST*(T15+T17)*T5)/RT2PI
      A=ASAV
      IF ( SC.LT.0. )  GO TO 5441
      QANS=PAP1X
      ANS=1.-QANS
      GO TO 5451
 5441 CONTINUE
      ANS=PAP1X
      QANS=1.-ANS
 5451 CONTINUE
      RETURN
C
C       THIS IS PART B1
C
 6011 CONTINUE
      N=0
      RTX=SQRT(X)
      T5=EXP (-X)
      RN=T5/(RTX*GAMPT5)
```

```
C
C        THIS IS PART B1 SUPPLEMENT
C
      IF ( X.GT.16.)  GO TO 6031
      T1=C0+RTX*(C1+RTX*(C2+RTX*(C3+RTX*(C4+RTX*(C5+RTX*(C6+C7*RTX)))))
     1 )
      T3=D0+RTX*(D1+RTX*(D2+RTX*(D3+RTX*(D4+RTX*(D5+RTX*(D6+RTX*D7)))))
     1 )
      DEL=(T5     *T1)/T3
      GO TO 6091
 6031 CONTINUE
      T=1./X
      T1=ALPH0+T*(ALPH1+T*(ALPH2+T*(ALPH3+T*ALPH4)))
      T3=BET0+T*(BET1+T*(BET2+T*(BET3+T*BET4)))
      DEL=(    T5   /RTX)*(1./RTPI +T1/(T3*X))
      GO TO 6091
 6071 CONTINUE
      N=1
      RN=EXP(-X)
      DEL=RN
 6091 CONTINUE
      PNCX=DEL
      CEE=AF+FLOAT(N)-1.
 6101 CONTINUE
      IF ( N.EQ.I )  GO TO 6111
      N=N+1
      CEE=CEE+1.
      RN=(X*RN)/CEE
      PNCX=PNCX+RN
      GO TO 6101
 6111 CONTINUE
      R=X*RN
      QANS=PNCX
      ANS=1.-QANS
      RETURN
C
C        THIS IS PART C
C
 7011 CONTINUE
      A2NM1=1.
      A2N=1.
      B2NM1=X
      B2N=X+1.-A
      AN=1.
 7031 CONTINUE
      A2NM1=X*A2N+AN*A2NM1
      B2NM1=X*B2N+AN*B2NM1
      AM0=A2NM1/B2NM1
      AN=AN+1.
      SA2N=AN-A
      A2N=A2NM1+SA2N*A2N
      B2N=B2NM1+SA2N*B2N
      AN0=A2N/B2N
      IF ( ABS(AN0-AM0).GE.(ACC*AN0) )  GO TO 7031
      QANS=R*AN0
```

```
      ANS=1.-QANS
      RETURN
C
C        THIS IS PART D
C
 9011 CONTINUE
      SNP1=1.
      SUM=SNP1
      CEE=A
 9031 CONTINUE
      CEE=CEE-1.
      SNP1=(SNP1*CEE)/X
      SUM=SUM+SNP1
      IF ( ABS(SNP1).GE.ACC )  GO TO 9031
      QANS=(R*SUM)/X
      ANS=1.-QANS
      RETURN
      END
```

```
      SUBROUTINE GAMCO3 ( A,H,     AC ,T1 )
      DIMENSION C(30)
      DATA ( C(I),I=3,18 ) /
     1  +.57721 56649 01533 E+0 ,
     2  -.65587 80715 20254 E+0 ,
     3  -.42002 63503 40952 E-1 ,
     4  +.16653 86113 82291 E+0 ,
     5  -.42197 73455 55443 E-1 ,
     6  -.96219 71527 87697 E-2 ,
     7  +.72189 43246 66310 E-2 ,
     8  -.11651 67591 85907 E-2,
     9  -.21524 16741 14951E-3,
     1  +.12805 02823 88116 E-3,
     2  -.20134 85478 07892 E-4 ,
     3  -.12504 93482 14267 E-5 ,
     4  +.11330 27231 98170 E-5 ,
     5  -.20563 38416 97761 E-6 ,
     6  +.61160 95104 49142 E-8 ,
     7  +.50020 07644 46922 E-8 /
      ACC=.5*AC
      I=INT(A)
      AF=A-FLOAT(I)
      IF ( AF.EQ.0 . )   GO TO 3091
      CEE=AF-1.
      IF ( AF.LT.0.5 )   CEE=AF
      PHI=CEE
      K=3
      ANK=C(K)*CEE
      SUM=ANK
 3011 CONTINUE
      CEE=CEE*PHI
      K=K+1
      ANK=C(K)*CEE
      SUM=SUM+ANK
      IF ( ABS(ANK).LT.(ACC*ABS(SUM) ) )   GO TO 3031
      IF ( K.EQ.18 )   GO TO 3031
      GO TO 3011
 3031 CONTINUE
      H=SUM
      IF ( AF.GE.0.5 )   H=(1.-A+SUM)/A
      T1=1.+SUM
      IF ( AF.LT.0.5 )   T1=AF*T1
      T1=1./T1
      IF ( I.EQ.0 )   RETURN
      AJ=AF-1.
      J=0
 3051 CONTINUE
      J=J+1
      AJ=AJ+1.
      T1=AJ*T1
      IF ( J.LT.I      )   GO TO 3051
 3071 CONTINUE
      GO TO 3151
 3091 CONTINUE
      J=0
```

```
      T1=1.
      IF ( I.EQ.1 )  RETURN
      I=I-1
 3131 CONTINUE
      J=J+1
      T1=FLOAT(J)*T1
      IF ( J.LT.I )  GO TO 3131
 3151 CONTINUE
      RETURN
      END
```